

ON THE IRREDUCIBLE REPRESENTATIONS OF A CLASS OF POINTED HOPF ALGEBRAS

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ABSTRACT. We parameterize the finite-dimensional irreducible representations of a class of pointed Hopf algebras over an algebraically closed field of characteristic zero by dominant characters. The Hopf algebras we are considering arise in the work of N. Andruskiewitsch and the second author. Special cases are the multiparameter deformations of the enveloping algebras of semisimple Lie algebras where the deforming parameters are not roots of unity and some of their finite-dimensional versions in the root of unity case.

INTRODUCTION

In this paper we parameterize the finite-dimensional irreducible representations of pointed Hopf algebras over an algebraically closed field k of characteristic zero which arise in recent classification work [2, 4]. These Hopf algebras are quotients of two-cocycle twists

$$H = (U \otimes A)^\sigma,$$

where U and A have the form $B \# k[G]$ where $k[G]$ is the group algebra of an abelian group G over k and B is a left $k[G]$ -module algebra and a left $k[G]$ -comodule coalgebra [12]. More precisely $B = \mathfrak{B}(X)$ is the Nichols algebra of a finite-dimensional Yetter-Drinfeld module over the group algebra $k[G]$. See [3, Section 2] for a discussion of Nichols algebras and Yetter-Drinfeld modules in general.

Thus

$$U = \mathfrak{B}(W) \# k[\Lambda] \quad \text{and} \quad A = \mathfrak{B}(V) \# k[\Gamma],$$

where Λ and Γ are abelian groups, and W and V are Yetter-Drinfeld modules over Λ and Γ respectively. We assume that W and V are direct sums of one-dimensional Yetter-Drinfeld modules. The 2-cocycle $\sigma : (U \otimes A) \otimes (U \otimes A) \rightarrow k$ is given in terms of a bilinear form $\beta : W \otimes V \rightarrow k$.

In the Sections 1 and 2 we study the set $\text{Irr}(H)$ of isomorphism classes of finite-dimensional irreducible H -modules. We assume that the finite-dimensional irreducible U - and A -modules are one-dimensional; by

Theorem 1.1 this assumption is satisfied in the infinite-dimensional generic case, that is, when the diagonal elements of the braiding matrix of W are not roots of unity, and when U and A are finite-dimensional. Then we have shown in [14] that the finite-dimensional irreducible H -modules have the form $L(\rho, \chi)$, where $\rho \in \widehat{\Lambda}$ and $\chi \in \widehat{\Gamma}$ are certain characters. We conclude from results of [13] and [14] that there are Yetter-Drinfeld submodules $W' \subset W$ and $V' \subset V$, and a Hopf algebra projection $H \rightarrow H'$ defining a bijection

$$(0.1) \quad \text{Irr}(H') \xrightarrow{\cong} \text{Irr}(H),$$

where $H' = (\mathfrak{B}(W') \# k[\Lambda] \otimes \mathfrak{B}(V') \# k[\Gamma])^{\sigma'}$, and σ' is the restriction of σ such that the restricted bilinear form $\beta' : W' \otimes V' \rightarrow k$ is *non-degenerate*.

In Section 2 we assume that the bilinear form β is non-degenerate. The main result of this section is Theorem 2.8 where we show that

$$(0.2) \quad \{(\rho, \chi) \in \widehat{\Lambda} \times \widehat{\Gamma} \mid (\rho, \chi) \text{ dominant}\} \rightarrow \text{Irr}((U \otimes A)^\sigma),$$

given by mapping (ρ, χ) onto the isomorphism class of $L(\rho, \chi)$, is bijective. The assumption in Theorem 2.8 is that the braiding matrix of W is generic, that is all its diagonal entries are not roots of unity, and of finite Cartan type [1]. The notion of a *dominant pair of characters* (ρ, χ) is defined in (2.16).

In Section 3 we apply the results of the first two sections to the class of pointed Hopf algebras studied in [4], in particular to the Hopf algebras of the type $U(\mathcal{D}, \lambda)$ or $u(\mathcal{D}, \lambda)$ described below. Let $(g_i)_{1 \leq i \leq \theta}$ be elements in Γ , let $(\chi_i)_{1 \leq i \leq \theta}$ be elements of $\widehat{\Gamma}$, the k -valued characters of Γ , and let $(a_{ij})_{1 \leq i, j \leq \theta}$ be a Cartan matrix of finite type. Then the collection

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i), (\chi_i), (a_{ij}))$$

is called a datum of finite Cartan type [4] if

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad q_{ii} \neq 1, \quad \text{with } q_{ij} = \chi_j(g_i) \text{ for all } 1 \leq i, j \leq \theta.$$

\mathcal{D} is called generic if no q_{ii} is a root of unity.

Let $\lambda = (\lambda_{ij})_{1 \leq i, j \leq \theta, i \not\sim j}$ be a family of linking parameters for \mathcal{D} , that is, $\lambda_{ij} \in k$ for all $1 \leq i, j \leq \theta, i \not\sim j$, $\lambda_{ji} = -q_{ji}\lambda_{ij}$, and if $g_i g_j = 1$ or $\chi_i \chi_j \neq \varepsilon$, then $\lambda_{ij} = 0$.

The Hopf algebra $U(\mathcal{D}, \lambda)$ is generated as an algebra by the group Γ , that is, by generators of Γ satisfying the relations of the group, and

x_1, \dots, x_θ , with the relations:

$$\begin{aligned} (\text{Action of the group}) \quad & gx_i g^{-1} = \chi_i(g) x_i, \text{ for all } i, \text{ and all } g \in \Gamma, \\ (\text{Serre relations}) \quad & \text{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \text{ for all } i \neq j, i \sim j, \\ (\text{Linking relations}) \quad & \text{ad}_c(x_i)(x_j) = \lambda_{ij}(1 - g_i g_j), \text{ for all } i \approx j. \end{aligned}$$

The coalgebra structure is given by

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad \Delta(g) = g \otimes g, \text{ for all } 1 \leq i \leq \theta, g \in \Gamma.$$

In [4] a glueing process was used to build $U(\mathcal{D}, \lambda)$ inductively by adding one connected component at a time and modding out some central group-like elements in each step. The methods of the present paper to parameterize the finite-dimensional irreducible modules do not apply to this description of $U(\mathcal{D}, \lambda)$, since in Theorem 2.8 we have to assume that the finite-dimensional irreducible modules over U and A are one-dimensional. In Lemma 3.2 we show that the linking graph of (\mathcal{D}, λ) in the generic case has no odd cycles. Hence the linking graph is bipartite, and we can give another description of $U(\mathcal{D}, \lambda)$ by one glueing only. As a consequence Theorem 2.8 applies and in Theorem 3.7 we obtain a bijection

$$(0.3) \quad \{\chi \in \widehat{\Gamma} \mid \chi \text{ dominant}\} \rightarrow \text{Irr}(U(\mathcal{D}, \lambda)),$$

where dominant characters χ are defined in (3.16). In the proof we use the reduction to the non-degenerate case by (0.1).

The Hopf algebras $U(\mathcal{D}, \lambda)$ with generic braiding matrix form a very general class of pointed Hopf algebras of finite Gelfand-Kirillov dimension and with abelian group of group-like elements. In Section 3.3 we introduce reduced data \mathcal{D}_{red} of finite Cartan type and define Hopf algebras $U(\mathcal{D}_{red}, l)$, where $l = (l_i)_{1 \leq i \leq n}$ is a family of non-zero scalars. The Hopf algebras $U(\mathcal{D}_{red}, l)$ are a reformulation of $U(\mathcal{D}, \lambda)$ in the non-degenerate case, that is, when any vertex of $\{1, \dots, \theta\}$ is linked to some other vertex. Thus to study the finite-dimensional irreducible $U(\mathcal{D}, \lambda)$ -modules it suffices to consider modules over $U(\mathcal{D}_{red}, l)$.

In Lemma 3.13 we describe $U(\mathcal{D}_{red}, l)$ by the usual generators E_i and $F_i, 1 \leq i \leq n$, and deformed Serre relations. In the end of Section 3.3 we note that the one-parameter deformation $U_q(\mathfrak{g})$ of $U(\mathfrak{g})$, where \mathfrak{g} is a semisimple Lie algebra (see [11]), Lusztig's version of the one-parameter deformation with more general group-like elements in [9], and the two-parameter deformations of $U(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_n)$ discussed in [6] are all special cases of $U(\mathcal{D}_{red}, l)$. In all these cases the parametrization of the finite-dimensional irreducible representations (of "type 1") by dominant weights is a special case of (0.3).

Our results on parameterization of the irreducible representations can be partially extended to the finite-dimensional versions of $U(\mathcal{D}, \lambda)$ in [5], where Γ is a finite abelian group. We assume that the root vector relations in [5] are all 0. The finite-dimensional Hopf algebras $u(\mathcal{D}, \lambda)$ defined in Section 3.4 generalize Lusztig's Frobenius kernels. The linking graph of $u(\mathcal{D}, \lambda)$ is bipartite if the Cartan matrix is simply laced but not in general. The finite-dimensional version of Theorem 2.8 applies to $u(\mathcal{D}, \lambda)$ with bipartite linking graph, and in Theorem 3.20 we obtain a bijection

$$(0.4) \quad \widehat{\Gamma} \rightarrow \text{Irr}(u(\mathcal{D}, \lambda)).$$

Throughout this paper k is an algebraically closed field of characteristic zero.

1. PRELIMINARIES AND GENERAL THEOREMS

We recall and reformulate some general results from [13] and [14] for the class of Hopf algebras we are considering. We refer to [3, Section 2] for a discussion of braided Hopf algebras, Nichols algebras and Yetter-Drinfeld modules in general, and to [16] and [10] for Hopf algebra theory.

Let Γ be an abelian group. A Yetter-Drinfeld module over $k[\Gamma]$ can be described as a Γ -graded vector space which is a Γ -module such that all g -homogeneous components, where $g \in \Gamma$, are stable under the Γ -action. We denote the category of Yetter-Drinfeld modules over $k[\Gamma]$ by ${}^{k[\Gamma]}_{k[\Gamma]}\mathcal{YD} = {}^\Gamma_\Gamma\mathcal{YD}$.

For $X \in {}^\Gamma_\Gamma\mathcal{YD}$, $g \in \Gamma$, and $\chi \in \widehat{\Gamma}$ we define

$$X_g = \{x \in X \mid \delta(x) = g \otimes x\}$$

and

$$X_g^\chi = \{x \in X_g \mid h \cdot x = \chi(h)x \text{ for all } h \in \Gamma\}.$$

The category ${}^\Gamma_\Gamma\mathcal{YD}$ is braided and for $X, Y \in {}^\Gamma_\Gamma\mathcal{YD}$

$$c : X \otimes Y \rightarrow Y \otimes X, x \otimes y \mapsto g \cdot y \otimes x, x \in X_g, y \in Y,$$

defines the braiding on $X \otimes Y$.

Let $\theta \geq 1$, $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$, and

$$q_{ij} = \chi_j(g_i), 1 \leq i, j \leq \theta.$$

Let $X \in {}^\Gamma_\Gamma\mathcal{YD}$ be the object with basis $x_i \in X_{g_i}^{\chi_i}$, $1 \leq i \leq \theta$. The braiding on the vector space X is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, 1 \leq i, j \leq \theta.$$

A braided graded Hopf algebra

$$R = \oplus_{n \geq 0} R(n) \text{ in } {}^\Gamma \mathcal{YD}$$

is a *Nichols algebra* of X if $X \cong R(1)$ in ${}^\Gamma \mathcal{YD}$, R is connected, that is, $k \cong R(0)$, and if

- (1) $R(1)$ consists of all the primitive elements of R and
- (2) as an algebra R is generated by $R(1)$.

The Nichols algebra of X exists and is unique up to isomorphism. We denote the Nichols algebra of X by $\mathfrak{B}(X)$. As an algebra and coalgebra, $\mathfrak{B}(X)$ only depends on the braided vector space (X, c) . The structure of $\mathfrak{B}(X)$ is known if the braiding of X is related to a Cartan matrix of finite type in the following way.

Let $(a_{ij})_{1 \leq i, j \leq \theta}$ be a Cartan matrix of finite type. Then the collection

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$$

is called a *datum of finite Cartan type* and the *braiding of X is of finite Cartan type* [4] if

$$(1.1) \quad q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad q_{ii} \neq 1, \quad \text{where } q_{ij} = \chi_j(g_i) \text{ for all } 1 \leq i, j \leq \theta.$$

The smash product $\mathfrak{B}(X) \# k[\Gamma]$ is a Hopf algebra over the field k where

$$(1.2) \quad gx_i g^{-1} = \chi_i(g)x_i, \text{ and}$$

$$(1.3) \quad \Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad \Delta(g) = g \otimes g$$

for all $g \in \Gamma, 1 \leq i \leq \theta$. Here and in the following we identify $x \in \mathfrak{B}(X)$ with $x \# 1$, and for $h \in k[\Gamma]$ we identify h with $1 \# h$ in the smash product. Thus we write

$$xh = (x \# 1)(1 \# h) = x \# h, \text{ and } hx = (1 \# h)(1 \# b) = (h_{(1)} \cdot b)h_{(2)}.$$

Suppose that the braiding of X is of finite Cartan type with Cartan matrix (a_{ij}) . For all $1 \leq i, j \leq n$ we write $i \sim j$ if i and j are in the same connected component of the Dynkin diagram of (a_{ij}) . We remark that \sim is an equivalence relation and the connected components of $I = \{1, 2, \dots, \theta\}$ are its equivalence classes, the set of which we denote by \mathcal{X} .

By [5, Lemma 2.3] there are $d_i \in \{1, 2, 3\}, 1 \leq i \leq \theta$, and $q_J \in k$ for all $J \in \mathcal{X}$, such that

$$(1.4) \quad q_{ii} = q_J^{2d_i}, d_i a_{ij} = d_j a_{ji} \text{ for all } J \in \mathcal{X}, i, j \in J.$$

We define

$$(1.5) \quad q'_{ij} = \begin{cases} (q_J)^{d_i a_{ij}}, & \text{if } i, j \in J, J \in \mathcal{X}, \\ 0, & \text{if } i \not\sim j. \end{cases}$$

Then

$$(1.6) \quad q_{ij}q_{ji} = q'_{ij}q'_{ji}, q_{ii} = q'_{ii} \text{ for all } 1 \leq i, j \leq \theta.$$

Thus in the terminology of [4], the braiding of X is twist equivalent to the braiding of Drinfeld-Jimbo type given by (q'_{ij}) .

The datum \mathcal{D} and the braiding of X are called *generic* [4] if

$$(1.7) \quad q_{ii} \text{ is not a root of unity for all } 1 \leq i \leq n.$$

If \mathcal{D} is a generic Cartan datum of finite type, then it is shown in [4] using results of Lusztig [9] and Rosso [15] that

$$\mathfrak{B}(X) \cong k\langle x_1, \dots, x_\theta \mid (\text{ad}_c(x_i))^{1-a_{ij}}(x_j) = 0, 1 \leq i, j \leq \theta, i \neq j \rangle,$$

where ad_c denotes the braided adjoint action. Hence if $y = x_{j_1} \cdots x_{j_s}$, and $1 \leq i, j_1, \dots, j_s \leq \theta, s \geq 1$, then

$$(\text{ad}_c(x_i))(y) = xy - q_{ij_1} \cdots q_{ij_s} y x_i.$$

In particular, if \mathfrak{g} is the semisimple Lie algebra with Cartan matrix (a_{ij}) , $q \in k$ is not a root of unity, and the braiding is given by (1.5), where $q_J = q$ for all connected components J , then $\mathfrak{B}(X) \cong U_q^-(\mathfrak{g})$.

Theorem 1.1. *Let*

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$$

be a datum of finite Cartan type, and let $X \in {}^\Gamma \mathcal{YD}$ be the object with basis $x_i \in X_{g_i}^{\chi_i}, 1 \leq i \leq \theta$. Then all finite-dimensional simple $\mathfrak{B}(X) \# k[\Gamma]$ -modules are one-dimensional if

- (1) \mathcal{D} is generic, or
- (2) *The group Γ is finite, for all $1 \leq i \leq \theta$, $\text{ord}(q_{ii})$ is odd, and $\text{ord}(q_{ii})$ is prime to 3 if i is in a component G_2 of the Dynkin diagram of (a_{ij}) .*

PROOF: In Case (1) the theorem is a special case of [14, Theorem 5.6]. In Case (2), $\mathfrak{B}(X)$ is finite-dimensional by [1, Theorem 1.1], [5, Theorem 5.1]. Since $\mathfrak{B}(X)$ is \mathbb{N} -graded it follows that the augmentation ideal $\mathfrak{B}(X)^+$ is nilpotent. Let $U = \mathfrak{B}(X) \# k[\Gamma]$. Since

$$U\mathfrak{B}(X)^+ = \mathfrak{B}(X)^+U, \text{ and } U/U\mathfrak{B}(X)^+ \cong k[\Gamma],$$

the simple U -modules are simple $k[\Gamma]$ -modules, hence one-dimensional.

□

For the rest of this section we fix abelian groups Λ and Γ , integers $n, m \geq 1$, elements $z_1, \dots, z_n \in \Lambda$ and $g_1, \dots, g_m \in \Gamma$, and nontrivial characters $\eta_1, \dots, \eta_n \in \widehat{\Lambda}$ and $\chi_1, \dots, \chi_m \in \widehat{\Gamma}$.

Let $W \in {}^\Lambda \mathcal{YD}$ have basis $u_i \in W_{z_i}^{\eta_i}$, $1 \leq i \leq n$ and let $V \in {}^\Gamma \mathcal{YD}$ have basis $a_j \in V_{g_j}^{\chi_j}$, $1 \leq j \leq m$. Let $U = \mathfrak{B}(W) \# k[\Lambda]$ and $A = \mathfrak{B}(V) \# k[\Gamma]$. Note that the restriction maps

$$(1.8) \quad \text{Alg}(\mathfrak{B}(W) \# k[\Lambda], k) \rightarrow \widehat{\Lambda}, \quad \text{Alg}(\mathfrak{B}(V) \# k[\Gamma], k) \rightarrow \widehat{\Gamma},$$

are bijective since the characters η_i, χ_j are all non-trivial.

We denote the preimage of a character $\psi \in \widehat{\Lambda}$ or $\widehat{\Gamma}$ by $\widetilde{\psi}$. Thus if $\psi \in \widehat{\Gamma}$, then $\widetilde{\psi}(a_j) = 0$, and $\widetilde{\psi}(g) = \psi(g)$ for all $1 \leq j \leq m, g \in \Gamma$.

The following results hold in this general context; we do not assume that the braidings are of finite Cartan type.

The next theorem describes how to define Hopf algebras maps from U to A^{cop} , the cop-version of the Hopf dual of A . The proof is based on the universal properties of the Nichols algebra.

Theorem 1.2. *In addition to the above let $\varphi : \Lambda \rightarrow \widehat{\Gamma}$ be a group homomorphism, $s : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ be a function, and let $l_1, \dots, l_n \in k$. Let $I' = \{1 \leq i \leq n \mid l_i \neq 0\}$. Assume that for all $i \in I'$ and $z \in \Lambda$*

$$(1.9) \quad \varphi(z_i) = \chi_{s(i)}^{-1} \text{ and } \eta_i(z) = \varphi(z)(g_{s(i)}).$$

Then there is a Hopf algebra map $\Phi : U \rightarrow A^{\text{cop}}$ such that

$$\begin{aligned} \Phi(z) &= \widetilde{\varphi(z)} \text{ for all } z \in \Lambda, \\ \Phi(u_i)(g) &= 0 \text{ for all } 1 \leq i \leq n, g \in \Gamma, \\ \Phi(u_i)(a_j) &= \delta_{s(i),j} l_i \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

PROOF: This is shown in [13, Corollary 9.1]. \square

By [8] any Hopf algebra map $\Phi : U \rightarrow A^{\text{cop}}$ determines a convolution invertible map $\tau : U \otimes A \rightarrow k$ by

$$(1.10) \quad \tau(u, a) = \Phi(u)(a),$$

and a 2-cocycle $\sigma : (U \otimes A) \otimes (U \otimes A) \rightarrow k$ by

$$(1.11) \quad \sigma(u \otimes a, u' \otimes a') = \varepsilon(a) \tau(u', a) \varepsilon(a'),$$

for all $u, u' \in U, a, a' \in A$. Let

$$H = (U \otimes A)^\sigma.$$

Recall that $(U \otimes A)^\sigma$ is a Hopf algebra which coincides with $U \otimes A$ as a coalgebra with componentwise comultiplication and whose algebra structure is defined by

$$\begin{aligned} (1.12) \quad (u \otimes a)(u' \otimes a') &= \sigma(h_{(1)}, h'_{(1)}) h_{(2)} h'_{(2)} \sigma^{-1}(h_{(3)}, h'_{(3)}) \\ &= u \tau(u'_{(1)}, a_{(1)}) u'_{(2)} \otimes a_{(2)} \tau^{-1}(u'_{(3)}, a_{(3)}) a' \end{aligned}$$

for all $u, u' \in U, a, a' \in A$, and $h = u \otimes a, h' = u' \otimes a'$. Note that $\tau^{-1}(u, a) = \tau(S(u), a)$ for all $u \in U, a \in A$. We view U and A as subalgebras of H by the embeddings $u \mapsto u \otimes 1$ and $a \mapsto 1 \otimes a$. Since $(u \otimes 1)(1 \otimes a) = u \otimes a$ we have $ua = u \otimes a$ under our identifications of U and A as subalgebras of $U \otimes A$.

Our main concern are the irreducible finite-dimensional representations of H . Let $\rho \in \widehat{\Lambda}, \chi \in \widehat{\Gamma}$ be characters. We define the left H -module U_χ with underlying vector space U by

$$(1.13) \quad (u'a) \cdot_\chi u = u' \tau(u_{(1)}, a_{(1)}) u_{(2)} \tilde{\chi}(a_{(2)}) \tau^{-1}(u_{(3)}, a_{(3)})$$

for all $u, u' \in U, a \in A$. Thus $U_\chi \cong H \otimes_A k_\chi$, where k_χ is the one-dimensional A -module defined by $\tilde{\chi}$.

Let $I(\rho, \chi)$ be the largest H -submodule of U_χ which is contained in the kernel of $\tilde{\rho} : U = \mathfrak{B}(W) \# k[\Lambda] \rightarrow k$. We define

$$L(\rho, \chi) = U_\chi / I(\rho, \chi).$$

Then $L(\rho, \chi)$ is a cyclic left H -module with generator

$$m_{\rho, \chi} = \text{residue class of } 1.$$

We also need another description of $L(\rho, \chi)$ as a subspace of the dual A^* . We define the right H -module A_ρ with underlying vector space A by

$$(1.14) \quad a \cdot_\rho (ua') = \tau(u_{(1)}, a_{(1)}) \rho(u_{(2)}) a_{(2)} a' \tau^{-1}(u_{(3)}, a_{(3)})$$

for all $u \in U, a, a' \in A$. Then the dual $(A_\rho)^*$ is a left H -module by

$$(1.15) \quad ((ua') \cdot_\rho p)(a) = p(a \cdot_\rho (ua'))$$

for all $u \in U, a, a' \in A$ and $p \in A^*$.

Note that by (1.15)

$$(1.16) \quad u \cdot_\rho p = \Phi(u_{(1)}) \rho(u_{(2)}) p \Phi(S(u_{(3)})) \text{ for all } u \in U, p \in A^*.$$

By [14, part c) of Proposition 2.6] there is a left H -isomorphism

$$(1.17) \quad U_\chi / I(\rho, \chi) \cong U \cdot_\rho \tilde{\chi}, \text{ with } m_{\rho, \chi} \mapsto \tilde{\chi}.$$

For any algebra R we denote by $\text{Irr}(R)$ the set of isomorphism classes of all finite-dimensional left R -modules.

Theorem 1.3. *In the situation of Theorem 1.2 assume that all finite-dimensional simple U - and A -modules are one-dimensional. Let σ be the 2-cocycle determined by Φ and $H = (U \otimes A)^\sigma$.*

(1) *The map*

$$L_H : \{(\rho, \chi) \mid \rho \in \widehat{\Lambda}, \chi \in \widehat{\Gamma}, \dim(L(\rho, \chi)) < \infty\} \rightarrow \text{Irr}(H),$$

given by $(\rho, \chi) \mapsto [L(\rho, \chi)]$, is a bijection.

- (2) Let $\rho \in \widehat{\Lambda}$, $\chi \in \widehat{\Gamma}$ and let L be a finite-dimensional simple left H -module. Then $L \cong L(\rho, \chi)$ if and only if there is a non-zero element $m \in L$ such that $a \cdot m = \widetilde{\chi}(a)m$, $z \cdot m = \rho(z)m$ for all $a \in A, z \in \Lambda$.

PROOF: By (1.8) part (1) follows from Theorem 1.2 and [14, Theorem 4.1].

To prove (2), let $m = m_{\rho, \chi} \in L(\rho, \chi)$. It follows from the definition of U_χ that $a \cdot 1 = \widetilde{\chi}(a)1$, hence

$$(1.18) \quad a \cdot m = \widetilde{\chi}(a)m \text{ for all } a \in A.$$

By (1.16) we have $z \cdot_\rho \widetilde{\chi} = \Phi(z)\rho(z)\widetilde{\chi}\Phi(z^{-1})\rho(z) = \rho(z)\widetilde{\chi}$ for all $z \in \Lambda$, since $\text{Alg}(A, k)$ is commutative by (1.8). Hence by (1.17)

$$(1.19) \quad z \cdot m = \rho(z)m \text{ for all } z \in \Lambda.$$

We have shown in [14, Corollary 3.5,a)] that $L(\rho, \chi)$ has a unique one-dimensional A -submodule. Thus up to a non-zero scalar the element $m \in L(\rho, \chi)$ is uniquely determined by (1.18), and the claim follows from (1.18). \square

Let $\varphi : R \rightarrow S$ be any ring homomorphism. Then the restriction functor $\varphi^* : {}_S\mathcal{M} \rightarrow {}_R\mathcal{M}$ maps a left S -module M to itself thought of as an R -module via pullback along φ . For any algebra R we denote by $\text{Irr}(R)$ the set of isomorphism classes of all finite-dimensional left R -modules.

Theorem 1.4. *In the situation of Theorem 1.2 assume that all finite-dimensional simple U - and A -modules are one-dimensional and that the restriction of s to I' is injective. Let $V' \subset V$ and $W' \subset W$ be the Yetter–Drinfeld submodules with bases $a_{s(i)}, i \in I'$, and $u_i, i \in I'$. Let $U' = \mathfrak{B}(W') \# k\Lambda$, $A' = \mathfrak{B}(V') \# k\Gamma$ and $H' = (U' \otimes A')^{\sigma'}$, where σ' is the restriction of σ to $U' \otimes A'$. Then the projections $\pi_W : W \rightarrow W'$, $\pi_V : V \rightarrow V'$ define a surjective bialgebra map $F : H \rightarrow H'$ determined by $F|_W = \pi_W$, $F|_V = \pi_V$, $F|_\Gamma = \text{id}_\Gamma$, and $F|_\Lambda = \text{id}_\Lambda$. The restriction functor F^* defines a bijection*

$$(1.20) \quad F^* : \text{Irr}(H') \rightarrow \text{Irr}(H),$$

and the diagram

$$\begin{array}{ccc} \widehat{\Lambda} \times \widehat{\Gamma} & \xrightarrow{L_{H'}} & \text{Irr}(H') \\ & \searrow L_H & \downarrow F^* \\ & & \text{Irr}(H) \end{array}$$

commutes.

PROOF: The first part follows by [13, Corollary 9.2]. The diagram commutes by [14, Proposition 2.7] with $U' = \overline{U}$, $A' = \overline{A}$, $f = F|U$, and $g = F|A$. Hence F^* is bijective by Theorem 1.3 for H and H' . \square

Remark 1.5. The restriction $s|I'$ of the previous theorem is injective when the braiding matrix $(q_{ij} = \eta_j(z_i))$ of W satisfies the following condition

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} \text{ for all } i, j,$$

where (a_{ij}) is a Cartan matrix of finite type and for all i the order of q_{ii} is greater than 3. \square

The remark is easily justified. By (1.9) we have

$$q_{ij} = \eta_j(z_i) = \varphi(z_i)(g_{s(j)}) = \chi_{s(i)}^{-1}(g_{s(j)})$$

for all $i \in I'$ and $1 \leq j \leq \theta$.

Assume $s(j) = s(l)$, where $j, l \in I'$. Then $q_{ij} = q_{il}$ and $q_{ji} = q_{li}$ for all $i \in I'$. Thus $q_{ii}^{a_{ij}} = q_{ij}q_{ji} = q_{il}q_{li} = q_{ii}^{a_{il}}$. Since $|a_{ij} - a_{il}| \leq 3$ and the order of q_{ii} is larger than 3 we have $a_{ij} = a_{il}$ for all $i \in I'$. Since $(a_{rs})_{r,s \in I'}$ is invertible necessarily $j = l$.

The results of Section 9 of [13] are given in terms of the bilinear form $\beta : W \otimes V \rightarrow k$ defined by $\beta(u_i \otimes a_j) = l_i \delta_{s(i)j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. In the preceding theorem, this form for $W' \otimes V'$ is $\beta' = \beta|(W' \otimes V')$ and is non-degenerate. Most applications for us will be in the context of Theorem 1.2 and the map s will be injective by virtue of Remark 1.5. In light of Theorem 1.4 to study the $L(\rho, \chi)$'s we may assume that β is non-degenerate.

2. THE MODULES $L(\rho, \chi)$ IN THE NON-DEGENERATE CASE

We continue with the notation following Theorem 1.1 and in the context of Theorem 1.2, imposing further restrictions on the hypothesis. We assume that β is non-degenerate, in particular that the map s is bijective. Without restriction we assume that $s = \text{id}$.

Let $\varphi : \Lambda \rightarrow \widehat{\Gamma}$ be a group homomorphism, and $l_1, \dots, l_n \neq 0$. By rescaling the generators a_i we could further assume that $l_i = 1$ for all $1 \leq i \leq n$; however this might not be convenient for computations.

As a result of our assumptions

$$(2.1) \quad \varphi(z_i) = \chi_i^{-1} \text{ and } \eta_i(z) = \varphi(z)(g_i) \text{ for all } 1 \leq i \leq n, z \in \Lambda,$$

and the Hopf algebra map $\Phi : U \rightarrow A^{\text{cop}}$ is given by

$$(2.2) \quad \Phi(z) = \widetilde{\varphi(z)}, \quad \Phi(u_i) = \delta_i \text{ for all } z \in \Lambda, 1 \leq i \leq n,$$

where $\delta_i : A \rightarrow k$ is the $(\varepsilon, \widetilde{\chi_i^{-1}})$ -derivation determined by

$$(2.3) \quad \delta_i(g) = 0, \quad \delta_i(a_j) = \delta_{ij}l_i \text{ for all } g \in \Gamma, 1 \leq i, j \leq n.$$

Remark 2.1. The Hopf algebra map $\Phi : U \rightarrow A^{op}$ is injective since it is injective on the primitive elements $P(\mathfrak{B}(W)) = W$ by our assumption that $l_i \neq 0$ for all i (see [16, Lemma 11.01]). \square

Let $H = (U \otimes A)^\sigma$, where σ is the 2-cocycle defined by τ with $\tau(u, a) = \Phi(u)(a)$ for all $u \in U, a \in A$. We write ua instead of $u \otimes a \in H$ for all $u \in U, a \in A$.

Let $\rho \in \widehat{\Lambda}$ and $\chi \in \widehat{\Gamma}$. We recall that $L(\rho, \chi) = U_\chi / I(\rho, \chi)$ is a cyclic left H -module and left U -module with generator

$$m = m_{\rho, \chi} = \text{residue class of } 1.$$

We denote the H -action on the quotient $L(\rho, \chi)$ by \cdot and we write $um = u \cdot m$ for all $u \in U$. Recall that $u \cdot_\chi u' = uu'$ for all $u, u' \in U$ by (1.13). Therefore for all $u \in U$, um is the residue class of u in $L(\rho, \chi)$ and the H -action on um is given for all $u' \in U$ and $a \in A$ by

$$(2.4) \quad u' \cdot um = u'um, \quad a \cdot um = (a \cdot_\chi u)m.$$

The latter holds since $a \cdot_\chi (u \cdot_\chi 1) = a \cdot_\chi u = (a \cdot_\chi u) \cdot_\chi 1$.

We denote the braiding matrix of W by (q_{ij}) , where

$$(2.5) \quad q_{ij} = \eta_j(z_i) \text{ for all } 1 \leq i, j \leq n.$$

Proposition 2.2. *Let $\rho \in \widehat{\Lambda}$ and $\chi \in \widehat{\Gamma}$. Then $L(\rho, \chi)$ is the k -span of all*

$$u_{i_1} \cdots u_{i_t} m, \text{ where } 1 \leq i_1, \dots, i_t \leq n \text{ and } t \geq 0.$$

The H -action on these elements is given for all $z \in \Lambda, g \in \Gamma$, and $1 \leq j \leq n$ by

- a) $z \cdot u_{i_1} \cdots u_{i_t} m = (\eta_{i_1} \cdots \eta_{i_t} \rho)(z) u_{i_1} \cdots u_{i_t} m,$
- b) $g \cdot u_{i_1} \cdots u_{i_t} m = (\chi_{i_1}^{-1} \cdots \chi_{i_t}^{-1} \chi)(g) u_{i_1} \cdots u_{i_t} m,$
- c) $u_j \cdot u_{i_1} \cdots u_{i_t} m = u_j u_{i_1} \cdots u_{i_t} m,$
- d) $a_j \cdot u_{i_1} \cdots u_{i_t} m = \sum_{l=1}^t \alpha_l(j, i_1, \dots, i_t) u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t} m,$

with $\alpha_l(j, i_1, \dots, i_t) = \delta_{ij} l_j \prod_{r=1}^{l-1} q_{i_r j} (1 - \prod_{s=l+1}^t q_{i_s j} q_{j i_s} \rho(z_j) \chi(g_j))$.

PROOF: We have seen in the proof of Theorem 1.2 that

$$z \cdot m = \rho(z)m.$$

Since $zu_i = \eta_i(z)u_i z$ for all $1 \leq i \leq n$, part a) follows by the first equation of (2.4). Note that part a) implies that the elements $u_{i_1} \cdots u_{i_t} m$ span $L(\rho, \chi)$.

Let $u = u_{i_1} \cdots u_{i_t}$. Note that by (1.3)

$$\begin{aligned}
 \Delta^2(u) &= u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \\
 (2.6) \quad &= (z_{i_1} \otimes z_{i_1} \otimes u_{i_1} + z_{i_1} \otimes u_{i_1} \otimes 1 + u_{i_1} \otimes 1 \otimes 1) \cdots \\
 &\quad \cdots (z_{i_t} \otimes z_{i_t} \otimes u_{i_t} + z_{i_t} \otimes u_{i_t} \otimes 1 + u_{i_t} \otimes 1 \otimes 1).
 \end{aligned}$$

By (2.4) $g \cdot um = (g \cdot_\chi u)m$, and by (1.13) and (2.6)

$$\begin{aligned}
 g \cdot_\chi u &= \Phi(u_{(1)})(g)u_{(2)}\chi(g)\Phi(u_{(3)})(g^{-1}) \\
 &= \varphi(z_{i_1} \cdots z_{i_t})(g)u_{i_1} \cdots u_{i_t}\chi(g),
 \end{aligned}$$

since $\Phi(u_{(1)})(g) = 0$ resp. $\Phi(u_{(3)})(g^{-1}) = 0$ if the term $u_{(1)}$ resp. $u_{(3)}$ contains a factor u_{i_l} . This proves part b) since

$$\varphi(z_{i_1} \cdots z_{i_t})(g) = (\chi_{i_1}^{-1} \cdots \chi_{i_t}^{-1})(g)$$

by (2.1).

Part c) is trivial, and to prove part d) we compute $a_j \cdot_\chi u$. Since $\Delta^2(a_j) = g_j \otimes g_j \otimes a_j + g_j \otimes a_j \otimes 1 + a_j \otimes 1 \otimes 1$, and $\chi(a_j) = 0$ it follows from (1.13) that

$$(2.7) \quad a_j \cdot_\chi u = \Phi(u_{(1)})(g_j)u_{(2)}\chi(g_j)\Phi(u_{(3)})(-a_j g_j^{-1}) + \Phi(u_{(1)})(a_j)u_{(2)}$$

as $S^{-1}(a_j) = -a_j g_j^{-1}$.

To compute the first term in (2.7) we first note that

$$(2.8) \quad \Phi(u_{(1)})(g_j)u_{(2)} = q_{i_1 j} \cdots q_{i_t j} u.$$

For by part a) of Theorem 1.2 we have $\Phi(u_{(1)})(g_j) = 0$ if $u_{(1)}$ contains at least one factor u_i , hence $\Phi(u_{(1)})(g_j)u_{(2)} = \Phi(z_{i_1} \cdots z_{i_t})(g_j)u$. Next we see that

$$(2.9) \quad u_{(1)}\Phi(u_{(2)})(a_j g_j^{-1}) = \sum_{l=1}^t u_{i_1} \cdots u_{i_{l-1}} z_{i_l} u_{i_{l+1}} \cdots u_{i_t} \Phi(u_{i_l})(a_j g_j^{-1})$$

since $\Phi(u_{(2)})(a_j g_j^{-1}) = 0$, if $u_{(2)}$ contains no or at least two factors u_{i_l} . Using (1.2), (2.3) and (2.5) we obtain from (2.9)

$$\begin{aligned}
 (2.10) \quad u_{(1)}\Phi(u_{(2)})(a_j g_j^{-1}) &= \\
 &\sum_{l=1}^t \delta_{i_l j} l_j q_{j j}^{-1} q_{i_l i_{l+1}} \cdots q_{i_l i_t} u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t} z_{i_l}.
 \end{aligned}$$

We now use (2.10) in Δ applied to (2.8) and obtain for the first term in (2.7)

$$(2.11) \quad \Phi(u_{(1)})(g_j)u_{(2)}\chi(g_j)\Phi(u_{(3)})(-a_jg_j^{-1}) = \\ -q_{i_1j} \cdots q_{i_tj}\chi(g_j) \sum_{l=1}^t \delta_{ijl} l_j q_{jj}^{-1} q_{i_l i_{l+1}} \cdots q_{i_l i_t} u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t} z_{i_l}.$$

Similarly we compute the second term in (2.7)

$$(2.12) \quad \Phi(u_{(1)})(a_j)u_{(2)} = \sum_{l=1}^t \delta_{ijl} l_j q_{i_1 i_l} \cdots q_{i_{l-1} i_l} u_{i_1} \cdots u_{i_{l-1}} u_{i_{l+1}} \cdots u_{i_t}.$$

Finally (2.7), (2.11) and (2.12) prove part d) in view of (2.4) and part a). \square

The presentation of $L(\rho, \chi)$ in (1.17) as the subspace $U \cdot_\rho \chi$ of A^* is helpful to actually calculate the elements of $L(\rho, \chi)$ as linear functions on A . The next proposition in principle describes an inductive procedure to calculate $L(\rho, \chi)$.

Proposition 2.3. *Let $\rho \in \widehat{\Lambda}$ and $\chi \in \widehat{\Gamma}$, and $1 \leq i, i_1, \dots, i_t \leq n$, where $t \geq 1$. Define $u = u_{i_1} \cdots u_{i_t}$. Then*

- a) $u_i \cdot_\rho \widetilde{\chi} = (1 - \rho(z_i)\chi(g_i))\Phi(u_i)\widetilde{\chi}.$
- b) $u_i \cdot_\rho (\Phi(u)\widetilde{\chi}) = \Phi([u_i u - \prod_{r=1}^t q_{ii_r} \rho(z_i)\chi(g_i) u u_{i_r}])\widetilde{\chi}.$
- c) $u_i^t \cdot_\rho \widetilde{\chi} = \prod_{l=0}^{t-1} (1 - q_{ii}^l \rho(z_i)\chi(g_i)) \Phi(u_i^t)\widetilde{\chi}.$

PROOF: Since $\Delta^2(u_i) = z_i \otimes z_i \otimes u_i + z_i \otimes u_i \otimes 1 + u_i \otimes 1 \otimes 1$, and $\rho(u_i) = 0, S(u_i) = -z_i^{-1}u_i$ it follows from (1.16) that for any $p \in A^*$

$$(2.13) \quad u_i \cdot_\rho p = \Phi(z_i)\rho(z_i)p\Phi(-z_i^{-1}u_i) + \Phi(u_i)p.$$

In particular

$$(2.14) \quad u_i \cdot_\rho (\Phi(u)\widetilde{\chi}) = \Phi(u_i)\Phi(u)\widetilde{\chi} - \rho(z_i)\Phi(z_i)\Phi(u)\widetilde{\chi}\Phi(z_i^{-1})\Phi(u_i) \\ = \Phi(u_i u)\widetilde{\chi} - q_{ii_1} \cdots q_{ii_t} \rho(z_i)\Phi(u)\widetilde{\chi}\Phi(u_i)$$

since, as $\text{Alg}(A, k)$ is abelian,

$$\begin{aligned} \Phi(z_i)\Phi(u)\widetilde{\chi}\Phi(z_i^{-1}) &= \Phi(z_i)\Phi(u)\Phi(z_i^{-1})\widetilde{\chi} \\ &= \Phi(z_i u z_i^{-1})\widetilde{\chi} \\ &= q_{ii_1} \cdots q_{ii_t} \Phi(u)\widetilde{\chi}. \end{aligned}$$

Since $\widetilde{\chi}\Phi(u_i)\widetilde{\chi}^{-1}$ and $\chi(g_i)\Phi(u_i)$ are both $(\varepsilon, \widetilde{\chi_i^{-1}})$ -derivations taking the same values on the generators $a_j, 1 \leq j \leq n$, and $g \in \Gamma$ of A we see that

$$(2.15) \quad \widetilde{\chi}\Phi(u_i)\widetilde{\chi}^{-1} = \chi(g_i)\Phi(u_i).$$

Part b) now follows from (2.14) and (2.15), and part a) is the special case of part b) with $t = 0$.

Part c) then follows by induction on t . The case $t = 1$ is part a), and the induction step is

$$\begin{aligned} u_i^{t+1} \cdot_\rho \tilde{\chi} &= u_i \cdot_\rho (u_i^t \cdot_\rho \tilde{\chi}) \\ &= \prod_{l=0}^{t-1} (1 - q_{ii}^l \rho(z_i) \chi(g_i)) u_i \cdot_\rho (\Phi(u_i^t) \tilde{\chi}) \\ &= \prod_{l=0}^t (1 - q_{ii}^l \rho(z_i) \chi(g_i)) \Phi(u_i^{t+1}) \tilde{\chi} \end{aligned}$$

since by part b) $u_i \cdot_\rho (\Phi(u_i^t) \tilde{\chi}) = (1 - q_{ii}^t \rho(z_i) \chi(g_i)) \Phi(u_i^{t+1}) \tilde{\chi}$. \square

In the next Corollary we formulate a necessary condition for the finiteness of the dimension of $L(\rho, \chi)$. We call a pair of characters $(\rho, \chi) \in \hat{\Lambda} \times \hat{\Gamma}$ *dominant* if there are natural numbers $m_i \geq 0$, $1 \leq i \leq n$, such that

$$(2.16) \quad q_{ii}^{m_i} \rho(z_i) \chi(g_i) = 1 \text{ for all } 1 \leq i \leq n.$$

Corollary 2.4. *Let $\rho \in \hat{\Lambda}$ and $\chi \in \hat{\Gamma}$ and assume that the braiding of W is generic. If $L(\rho, \chi)$ is finite-dimensional, then the pair (ρ, χ) is dominant.*

PROOF: Let $1 \leq i \leq n$. The elements $u_i^t m$, $t \geq 0$, of $L(\rho, \chi)$ are linearly dependent. By part a) of Proposition 2.2, the group Λ acts on $u_i^t m$ via the character $\eta_i^t \rho$. Since q_{ii} is a root of unity, $\eta_i^t \rho \neq \eta_i^{t'} \rho$ for all $t \neq t'$, there is an integer $m_i \geq 0$ such that $u_i^t m = 0$ for all $t > m_i$, and $u_i^t m \neq 0$ for all $0 \leq t \leq m_i$. It is well-known that $u_i^t \neq 0$ for all $t \geq 0$ since q_{ii} is not a root of unity (see for example [3, Example 2.9]). Since Φ is injective by Remark 2.1, it follows that $\Phi(u_i^t) \neq 0$ for all $t \geq 0$, and thus $q_{ii}^{m_i} \eta_i(z_i) \chi(g_i) = 1$ by part c) of Proposition 2.3 and (1.17). \square

As an example we consider the easiest case where U and A are quantum linear spaces, that is (2.17) holds.

Corollary 2.5. *Let $\rho \in \hat{\Lambda}$ and $\chi \in \hat{\Gamma}$ and assume that the braiding of W is generic and satisfies*

$$(2.17) \quad q_{ij} q_{ji} = 1 \text{ for all } 1 \leq i, j \leq n, i \neq j.$$

Then for all $t_1, \dots, t_n \geq 0$,

$$(u_1^{t_1} \cdots u_n^{t_n}) \cdot_\rho \tilde{\chi} = \prod_{i=1}^n \prod_{l_i=0}^{t_i-1} (1 - q_{ii}^{l_i} \rho(z_i) \chi(g_i)) \Phi(u_1^{t_1} \cdots u_n^{t_n}) \tilde{\chi}.$$

PROOF: It follows from (2.17) that

$$(2.18) \quad u_i u_j = q_{ij} u_j u_i \text{ for all } 1 \leq i, j \leq n, i \neq j,$$

since the skew-commutator $u_i u_j - q_{ij} u_j u_i$ is primitive in $\mathfrak{B}(W)$.

We prove by induction on $1 \leq j \leq n$ that

$$(u_j^{t_j} \cdots u_n^{t_n}) \cdot_\rho \tilde{\chi} = \prod_{i=j}^n \prod_{l_i=0}^{t_i-1} (1 - q_{ii}^{l_i} \rho(z_i) \chi(g_i)) \Phi(u_j^{t_j} \cdots u_n^{t_n}) \tilde{\chi}$$

for all $t_j, \dots, t_n \geq 0$.

The case $j = n$ is part c) of Proposition 2.3. Assume the formula for $j+1 \leq n$. Then we prove the claim for j by induction on t_j . If $t_j = 0$, then the formula is true by assumption, and for all $t_j \geq 0$ we see by induction on t_j that

$$(2.19) \quad \begin{aligned} (u_j^{t_j+1} \cdots u_n^{t_n}) \cdot_\rho \tilde{\chi} &= u_j \cdot_\rho \left((u_j^{t_j} \cdots u_n^{t_n}) \cdot_\rho \tilde{\chi} \right) \\ &= \prod_{i=j}^n \prod_{l_i=0}^{t_i-1} (1 - q_{ii}^{l_i} \rho(z_i) \chi(g_i)) u_j \cdot_\rho (\Phi(u_j^{t_j} \cdots u_n^{t_n}) \tilde{\chi}). \end{aligned}$$

By part b) of Proposition 2.3,

$$\begin{aligned} &u_j \cdot_\rho \left(\Phi(u_j^{t_j} \cdots u_n^{t_n}) \tilde{\chi} \right) \\ &= \left(\Phi(u_j^{t_j+1} \cdots u_n^{t_n}) - q_{jj}^{t_j} q_{j,j+1}^{t_{j+1}} \cdots q_{jn}^{t_n} \rho(z_j) \chi(g_j) \Phi(u_j^{t_j} \cdots u_n^{t_n} u_j) \right) \tilde{\chi} \\ &= \left(1 - q_{jj}^{t_j} \rho(z_j) \chi(g_j) \right) \Phi(u_j^{t_j+1} \cdots u_n^{t_n}) \tilde{\chi}, \end{aligned}$$

since $u_j^{t_j} \cdots u_n^{t_n} u_j = q_{j+1,j}^{t_{j+1}} \cdots q_{jn}^{t_n} u_j^{t_j+1} \cdots u_n^{t_n}$ by (2.18), and all the q_{jl} and q_{lj} except for $l = j$ cancel by (2.17). Hence the claim follows from (2.19). \square

Corollary 2.6. *Let $\rho \in \hat{\Lambda}$ and $\chi \in \hat{\Gamma}$ and assume that the braiding of W is generic and satisfies (2.17). Then $L(\rho, \chi)$ is finite-dimensional if and only if the pair (ρ, χ) is dominant. In this case, the elements $u_1^{t_1} \cdots u_n^{t_n} m$, $0 \leq t_i \leq m_i$ for all $1 \leq i \leq n$, form a basis of $L(\rho, \chi)$.*

PROOF: It is well-known that by (1.7) and (2.17) $\mathfrak{B}(W)$ is generated by u_1, \dots, u_n with defining relations

$$u_i u_j - q_{ij} u_j u_i = 0 \text{ for all } 1 \leq i, j \leq n, i \neq j,$$

and has PBW-basis $u_1^{t_1} \cdots u_n^{t_n}$, $t_1, \dots, t_n \geq 0$ (see for example [4, Theorem 4.2]).

If $L(\rho, \chi)$ is finite-dimensional, then (ρ, χ) is dominant by Corollary 2.4. Conversely, assume that (ρ, χ) is dominant. Since Φ is injective by

Remark 2.1, and the elements $u_1^{t_1} \cdots u_n^{t_n}, 0 \leq t_i \leq m_i$ for all $1 \leq i \leq n$, of the PBW-basis are linearly independent it follows from Corollary 2.5 that the elements $u_1^{t_1} \cdots u_n^{t_n} m, 0 \leq t_i \leq m_i$ for all $1 \leq i \leq n$, are linearly independent, hence a basis of $L(\rho, \chi)$ since $u_1^{t_1} \cdots u_n^{t_n} = 0$ if $t_i > m_i$ for one i . \square

We now extend the finiteness criterion for quantum linear spaces in Corollary 2.6 to braidings of finite Cartan type.

Theorem 2.7. *Assume that the braiding of W is generic and of finite Cartan type. let $e_1, \dots, e_n > 1$ be natural numbers. Then*

$$\mathfrak{B}(W)/(\sum_{1 \leq i \leq n} \mathfrak{B}(W)u_i^{e_i})$$

is finite-dimensional.

PROOF: As explained in Section 1 there are $d_i \in \{1, 2, 3\}, 1 \leq i \leq n$, and $q_J \in k, J \in \mathcal{X}$, such that

$$q_{ii} = q_J^{2d_i}, d_i a_{ij} = d_j a_{ji}$$

for all $J \in \mathcal{X}, i, j \in J$. We define (q'_{ij}) by (1.5).

Let $\mathbb{Z}[I]$ be the free abelian group of rank n with basis $\alpha_1, \dots, \alpha_n$. Define characters $\psi, \psi' \in \widehat{\mathbb{Z}[I]}$ by

$$(2.20) \quad \psi_j(\alpha_i) = q_{ij}, \psi'_j(\alpha_i) = q'_{ij} \text{ for all } 1 \leq i, j \leq n.$$

Let $X, X' \in {}_{\mathbb{Z}[I]}^{\mathbb{Z}[I]}\mathcal{YD}$ be Yetter-Drinfeld modules with bases $x_i \in X_{\alpha_i}^{\psi}$ and $x'_i \in X'_{\alpha_i}^{\psi'}, 1 \leq i \leq n$.

Then

$$(2.21) \quad \mathfrak{B}(W) \rightarrow \mathfrak{B}(X), u_i \mapsto x_i, 1 \leq i \leq n,$$

defines an isomorphism of algebras and coalgebras since the braiding matrices of W and X coincide. Hence it suffices to prove the theorem for X instead of W .

By (1.6) and [4, Proposition 2] there is a 2-cocycle

$$\sigma : \mathbb{Z}[I] \times \mathbb{Z}[I] \rightarrow k^\times$$

of the group $\mathbb{Z}[I]$ and a k -linear isomorphism

$$(2.22) \quad \Psi : \mathfrak{B}(X) \rightarrow \mathfrak{B}(X') \text{ with } \Psi(x_i) = x'_i, 1 \leq i \leq n$$

such that for all $\alpha, \beta \in \mathbb{Z}[I]$ and $x \in \mathfrak{B}(X)_\alpha, y \in \mathfrak{B}(X)_\beta$,

$$(2.23) \quad \Psi(xy) = \sigma(\alpha, \beta) \Psi(x) \Psi(y).$$

Let $\mathcal{X} = \{I_1, \dots, I_t\}$ be the set of connected components of $\{1, 2, \dots, n\}$. For all $1 \leq s \leq t$ we let $X_s = \bigoplus_{i \in I_s} kx_i$. Then the natural map

$$\mathfrak{B}(X_1) \otimes \cdots \otimes \mathfrak{B}(X_t) \rightarrow \mathfrak{B}(X)$$

given by multiplication is an isomorphism by [4, Lemma 1.4]. Since $x_i x_j = q_{ij} x_j x_i$ for all $1 \leq i, j \leq n, i \approx j$, it induces a surjective linear map from

$$\mathfrak{B}(X_1) / \left(\sum_{i \in I_1} \mathfrak{B}(X_1) x_i^{e_i} \right) \otimes \cdots \otimes \mathfrak{B}(X_t) / \left(\sum_{i \in I_t} \mathfrak{B}(X_t) x_i^{e_i} \right)$$

to

$$\mathfrak{B}(X) / \left(\sum_{1 \leq i \leq n} \mathfrak{B}(X) x_i^{e_i} \right).$$

Thus we have reduced the claim to the connected case.

If the Dynkin diagram of (a_{ij}) is connected, let $q = q_I$. Then by results of Lusztig [9, Section 37] and Rosso [15, Theorem 15]

$$\mathfrak{B}(X') = k \langle x'_1, \dots, x'_n \mid (\text{ad}_{c'}(x'_i))^{1-a_{ij}}(x'_j) = 0, i \neq j \rangle$$

is isomorphic to $U_q^-(\mathfrak{g})$, where \mathfrak{g} is the semisimple Lie algebra with Cartan matrix (a_{ij}) . The elements x'_i correspond to the F_i . Hence by [9, Proposition 6.3.4] or [11, Proposition 5.9]

$$\mathfrak{B}(X') / \left(\sum_{1 \leq i \leq n} \mathfrak{B}(X') x_i'^{e_i} \right)$$

is finite-dimensional. By (2.22) and (2.23), Ψ induces an isomorphism of vector spaces

$$\mathfrak{B}(X) / \left(\sum_{1 \leq i \leq n} \mathfrak{B}(X) x_i^{e_i} \right) \cong \mathfrak{B}(X') / \left(\sum_{1 \leq i \leq n} \mathfrak{B}(X') x_i'^{e_i} \right),$$

and the theorem is proved. \square

We can now prove the main result of this section.

Theorem 2.8. *Assume that the braiding of W is generic and of finite Cartan type. Then the map*

$$\{(\rho, \chi) \in \widehat{\Lambda} \times \widehat{\Gamma} \mid (\rho, \chi) \text{ dominant}\} \rightarrow \text{Irr}((U \otimes A)^\sigma),$$

given by $(\rho, \chi) \mapsto [L(\rho, \chi)]$, is bijective.

PROOF: Let $(\rho, \chi) \in \widehat{\Lambda} \times \widehat{\Gamma}$ be a dominant pair. By Theorem 1.1, Theorem 1.3 and Proposition 2.4 it suffices to show that $L(\rho, \chi)$ is finite-dimensional.

By definition $q_{ii}^{m_i} \rho(z_i) \chi(g_i) = 1$ for all $1 \leq i \leq n$, where $m_i \geq 0$ are natural numbers. Then it follows from Proposition 2.3 c) that $u_i^{m_i+1} \in I(\rho, \chi)$ for all $1 \leq i \leq n$. The natural map

$$\mathfrak{B}(W) \subset U \rightarrow L(\rho, \chi) = U/I(\rho, \chi)$$

is surjective, since $z \cdot m = \rho(z)m$ for all $z \in \Lambda$ by Proposition 2.2 a). Thus we have a surjective linear map

$$\mathfrak{B}(W) / \left(\sum_{1 \leq i \leq n} \mathfrak{B}(W) u_i^{m_i+1} \right) \rightarrow L(\rho, \chi),$$

and $L(\rho, \chi)$ is finite-dimensional by Theorem 2.7. \square

3. APPLICATION TO POINTED HOPF ALGEBRAS GIVEN BY A DATUM OF FINITE CARTAN TYPE AND LINKING PARAMETERS

Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$ be a datum of finite Cartan type and suppose that λ is a family of linking parameters for \mathcal{D} in the sense of [2]. We apply the theory developed in Sections 1 and 2 to the infinite-dimensional Hopf algebra $U(\mathcal{D}, \lambda)$ in general and to the finite-dimensional version $u(\mathcal{D}, \lambda)$ under some restrictions. See definitions 3.4 and 3.18.

Each is a quotient of an $H = (U \otimes A)^\sigma$ where the 2-cocycle σ is determined by the linking parameters. The finite-dimensional irreducible left modules for $U(\mathcal{D}, \lambda)$ are parameterized by a subset of $\widehat{\Gamma}$ and the irreducible modules for $u(\mathcal{D}, \lambda)$ by $\widehat{\Gamma}$; these irreducible modules arise from $L(\rho, \chi)$'s defined for H where χ determines ρ .

3.1. The Linking Graph. Let

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$$

be a datum of finite Cartan type. Note that by (1.1)

$$(3.1) \quad q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}} \text{ for all } 1 \leq i, j \leq \theta.$$

Let $1 \leq i, j \leq \theta$. Then $i \sim j$, that is, i and j lie in the same connected component of the Dynkin diagram of (a_{ij}) if and only if $i = j$ or there are indices $1 \leq i_1, \dots, i_k \leq \theta$, $i_l \neq i_{l+1}$ for all $1 \leq l < k$, such that $a_{i_l i_{l+1}} \neq 0$ for all $1 \leq l < k$. In this case we write

$$(3.2) \quad a(i, j) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}, b(i, j) = a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_k i_{k-1}},$$

and by (3.1) we have

$$(3.3) \quad q_{ii}^{a(i, j)} = q_{jj}^{b(i, j)}.$$

Note that $a(i, j)$ and $b(i, j)$ depend on the choice of the sequence i_1, \dots, i_k .

A family $\lambda = (\lambda_{ij})_{1 \leq i, j \leq \theta, i \not\sim j}$ of elements in k is called a *family of linking parameters for \mathcal{D}* if the following conditions are satisfied for all $1 \leq i, j \leq \theta, i \not\sim j$:

$$(3.4) \quad \text{If } g_i g_j = 1 \text{ or } \chi_i \chi_j \neq \varepsilon \text{ then } \lambda_{ij} = 0;$$

$$(3.5) \quad \lambda_{ji} = -q_{ji} \lambda_{ij}.$$

This definition is a formal extension of [2, Section 5.1] where λ_{ij} was only defined for $i < j$ and $i \not\sim j$. Note that by (3.5)

$$\lambda_{ij} = -q_{ji}^{-1} \lambda_{ji} = -q_{ij} \lambda_{ji}$$

since $q_{ij} q_{ji} = 1$ for all $i \not\sim j$. Note that (3.4) and (3.5) are met when $\lambda_{ij} = 0$ for all i, j . We let 0 denote this family of linking parameters.

Vertices $1 \leq i, j \leq \theta$ are called *linkable* if $i \not\sim j$, $g_i g_j \neq 1$ and $\chi_i \chi_j = \varepsilon$. Then (see [2, Section 5.1])

$$(3.6) \quad q_{ii} = q_{jj}^{-1} \text{ if } i, j \text{ are linkable.}$$

The next lemma is [2, Lemma 5.6].

Lemma 3.1. *Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$ be a datum of finite Cartan type, and assume that $\text{ord}(q_t) > 3$ for all $1 \leq t \leq \theta$.*

- (1) *If vertices i, k and j, l are linkable, then $a_{ij} = a_{kl}$.*
- (2) *A vertex i cannot be linkable to two different vertices j, k .*

We say that linkable vertices $1 \leq i, j \leq \theta$ are *linked* if $\lambda_{ij} \neq 0$.

Let λ be a family of linking parameters for a datum of Cartan type $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$. We define the *linking graph* of (\mathcal{D}, λ) as follows: The set of its vertices is \mathcal{X} , the set of all connected components of $I = \{1, \dots, n\}$. There is an edge between $J_1, J_2 \in \mathcal{X}$ if and only there are elements $i \in J_1, j \in J_2$ such that i, j are linked. Recall that a graph is called *bipartite* if the set of its vertices is the disjoint union of subsets X^+, X^- such that there is no edge between vertices in X^+ or in X^- .

Lemma 3.2. *Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$ be a datum of Cartan type, and λ a family of linking parameters for \mathcal{D} . Assume one of the following conditions:*

- (1) *The Cartan matrix (a_{ij}) is simply laced, that is $a_{ij} \in \{0, -1\}$ for all $1 \leq i, j \leq \theta, i \neq j$, and $\text{ord}(q_{ii})$ is odd for all $1 \leq i \leq \theta$.*
- (2) *For all $1 \leq i \leq \theta$, q_{ii} is not a root of unity.*

Then the linking graph of (\mathcal{D}, λ) is bipartite.

PROOF: A graph is bipartite if and only if it contains no cycle of odd length [7, Proposition 0.6.1]. Assume the linking graph has a cycle of odd length n . Then there are connected components J_1, \dots, J_n , and $i_l \in J_l, 1 \leq l \leq n$, and $j_l \in J_l, 2 \leq l \leq n+1, J_{n+1} = J_1$ with $\lambda_{i_l j_{l+1}} \neq 0$ for all $1 \leq l \leq n$. Let $i_{n+1} = i_1$. For simplicity we write $q_i = q_{ii}$ for all i . Since i_l and j_l are in the same connected component for all $2 \leq l \leq n+1$, by (3.3) $q_{i_l}^{a_l} = q_{j_l}^{b_l}$ for all $2 \leq l \leq n+1$, where $a_l = a(i_l, j_l)$ and $b_l = b(i_l, j_l)$ are chosen as in (3.2). By (3.6), $q_{i_l} = q_{j_{l+1}}^{-1}$ for all $1 \leq l \leq n$. Hence $q_{i_l}^{a_l} = q_{i_{l-1}}^{-b_l}$ for all $2 \leq l \leq n+1$, and

$$q_{i_{n+1}}^{a_{n+1} \cdots a_2} = q_{i_1}^{(-1)^n b_{n+1} \cdots b_2}.$$

Since $i_{n+1} = i_1$, and n is odd, the order of q_{i_1} must divide

$$a_{n+1} \cdots a_2 + b_{n+1} \cdots b_2.$$

This is impossible in both cases (1) and (2). \square

However, in general the linking graph of (\mathcal{D}, λ) is not necessarily bipartite.

Example 3.3. One can link an odd number of copies of B_2 in a circle, where the group Γ is \mathbb{Z}^{2n} or $(\mathbb{Z}/(N))^{2n}$ such that N is an integer dividing $1 + 2^n$, and where g_1, \dots, g_{2n} are the canonical basis elements of Γ . \square

3.2. The Infinite-dimensional Case. In this section we fix a generic Cartan datum of finite type $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$, and a family $\lambda = (\lambda_{ij})_{1 \leq i, j \leq \theta, i \neq j}$ of linking parameters for \mathcal{D} . We define $q_{ij} = \chi_j(g_i)$ for all $1 \leq i, j \leq \theta$.

Let $X \in {}^\Gamma \mathcal{YD}$ be the vector space with basis $x_i \in X_{g_i}^{\chi_i}, 1 \leq i \leq \theta$. The tensor algebra $T(X)$ is an algebra in ${}^\Gamma \mathcal{YD}$. Thus the smash product $T(X) \# k[\Gamma]$ is a biproduct and thus has a Hopf algebra structure. We identify $T(X)$ with the free algebra $k\langle x_1, \dots, x_\theta \rangle$.

Definition 3.4. Let $U(\mathcal{D}, \lambda)$ be the quotient Hopf algebra of the smash product $k\langle x_1, \dots, x_\theta \rangle \# k[\Gamma]$ modulo the ideal generated by

$$(3.7) \quad \text{ad}_c(x_i)^{1-a_{ij}}(x_j), \text{ for all } 1 \leq i, j \leq \theta, i \sim j, i \neq j,$$

$$(3.8) \quad x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(1 - g_i g_j), \text{ for all } 1 \leq i < j \leq \theta, i \not\sim j.$$

Here, the k -linear endomorphism $\text{ad}_c(x_i)$ of the free algebra is given for all y by the braided commutator $\text{ad}_c(x_i)(y) = x_i y - (g_i \cdot y) x_i$.

We denote the images of x_i and $g \in \Gamma$ in $U(\mathcal{D}, \lambda)$ again by x_i and g . Note that in $U(\mathcal{D}, \lambda)$

$$x_i x_j - q_{ij} x_j x_i = \lambda_{ij}(1 - g_i g_j), \text{ for all } 1 \leq i, j \leq \theta, i \not\sim j$$

by (3.5).

The elements in (3.7) and (3.8) are skew-primitive. Hence $U(\mathcal{D}, \lambda)$ is a Hopf algebra with

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad 1 \leq i \leq \theta.$$

Then $U(\mathcal{D}, 0)$ is the biproduct $\mathfrak{B}(X) \# k[\Gamma]$.

In the next Lemma we write down the relations (3.7) explicitly, and we give an equivalent formulation replacing the elements x_i by $x_i g_i^{-1}$. The elements x_i correspond to the usual E_i , and the elements $x_i g_i^{-1}$ to the F_i in $U_q(\mathfrak{g})$, \mathfrak{g} a semisimple Lie algebra. We recall the box notation for q -integers (see [9, 1.3], [11, Chapter 0]). Let v be an indeterminate. For all integers $n \geq 0$ let

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}},$$

$[0]! = 1$, and $[n]! = [1][2] \cdots [n]$ if $n \geq 1$. For all $0 \leq i \leq n$ define

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]!}{[i]![n-i]!}.$$

If $0 \neq q \in k$, then $[n]_q$ and $\begin{bmatrix} n \\ i \end{bmatrix}_q$ are the specializations at $v \mapsto q$.

Lemma 3.5. *Let $J \in \mathcal{X}$ be a connected component of $\{1, 2, \dots, \theta\}$, and define $F_i = x_i g_i^{-1}$ for all $i \in J$. Choose $q_J \in k$ such that (1.4) is satisfied. Then the following equalities hold in the smash product $k\langle x_1, \dots, x_\theta \rangle \# k[\Gamma]$ for all $i, j \in J$ and any natural number a :*

$$(3.9) \quad (\text{ad}_c x_i)^a(x_j) = \sum_{s=0}^a (-q_J^{d_i(a-1)} q_{ij})^s \begin{bmatrix} a \\ s \end{bmatrix}_{q_J^{d_i}} x_i^{a-s} x_j x_i^s,$$

$$(3.10) \quad (\text{ad}_c x_i)^a(x_j) = \sum_{s=0}^a (-q_J^{d_i(a-1)} q_{ji})^s \begin{bmatrix} a \\ s \end{bmatrix}_{q_J^{d_i}} F_i^{a-s} F_j F_i^s \\ \cdot g_i^a g_j (q_J^{d_i(a-1)} q_{ij})^a.$$

PROOF: We extend $\text{ad}_c x_i$ to a map on the smash product by

$$\text{ad}_c x_i = L_{x_i} - R_{x_i} \sigma_i,$$

where L_{x_i} and R_{x_i} denote left and right multiplication with x_i , and σ_i is the inner automorphism defined for all $y \in k\langle x_1, \dots, x_\theta \rangle \# k[\Gamma]$ by

$\sigma_i(y) = g_i y g_i^{-1}$. Since $(R_{x_i} \sigma_i) L_{x_i} = q_J^{2d_i} L_{x_i} (R_{x_i} \sigma_i)$, we can compute $(L_{x_i} - R_{x_i} \sigma_i)^a$ by the q -binomial formula [9, 1.3.5], and (3.9) follows from the equality $(R_{x_i} \sigma_i)^n = R_{x_i^n} \sigma_i^n q_J^{d_i n(n-1)}$ for all natural numbers n . Then (3.10) follows from (3.9) by a somewhat tedious computation where we write $x_i^{a-s} x_j x_i^s$ as a multiple of $F_i^{a-s} F_j F_i^s$. \square

Suppose that $\lambda \neq 0$. In [4] a glueing process was used to build $U(\mathcal{D}, \lambda)$ inductively by adding one connected component at a time and thus to obtain a basis of the algebra. The methods of the present paper to parameterize the finite-dimensional irreducible modules do not apply to this description of $U(\mathcal{D}, \lambda)$.

Since by Lemma 3.2 the linking graph of (\mathcal{D}, λ) is bipartite, we can give another description of $U(\mathcal{D}, \lambda)$ by one glueing only and show that the algebra is a quotient of $(U \otimes A)^\sigma$ where some central group-likes are identified with 1 and U and A are biproducts. As a consequence Theorem 2.8 applies.

In the inductive construction of [4] each stage yields a quotient of the form just described; however, U is not a biproduct and the finite-dimensional irreducible U -modules are not necessarily one-dimensional.

We now give our construction of $U(\mathcal{D}, \lambda)$. By Lemma 3.1 there are non-empty disjoint subsets $\mathcal{X}^-, \mathcal{X}^+ \subseteq \mathcal{X}$ with $\mathcal{X} = \mathcal{X}^- \cup \mathcal{X}^+$, an integer $n \geq 1$ and an injective map $t : \{1, \dots, 2n\} \rightarrow \{1, \dots, \theta\}$, such that

$$t(i) \in I^- := \bigcup_{J \in \mathcal{X}^-} J, \quad t(n+i) \in I^+ := \bigcup_{J \in \mathcal{X}^+} J,$$

for all $1 \leq i \leq n$, and such that $(t(i), t(n+i))$ and $(t(n+i), t(i))$, where $1 \leq i \leq n$ are all the linked pairs of elements in $\{1, \dots, \theta\}$.

Let Λ be the free abelian group with basis $z_i, i \in I^-$. For all $j \in I^-$ let $\eta_j \in \hat{\Lambda}$ be defined by $\eta_j(z_i) = \chi_j(g_i)$ for all $i \in I^-$.

Let $W \in {}^\Lambda \mathcal{YD}$ with basis $u_i \in W_{z_i}^{\eta_i}, i \in I^-$, and $V \in {}^\Gamma \mathcal{YD}$ with basis $a_j \in V_{g_j}^{\chi_j}, j \in I^+$. We define

$$U = \mathfrak{B}(W) \# k[\Lambda] \text{ and } A = \mathfrak{B}(V) \# k[\Gamma].$$

Theorem 3.6. *Assume the situation above. Then for all $i \in I^-$ there is a unique algebra map*

$$(3.11) \quad \gamma_i : A \rightarrow k \text{ with } \gamma_i(a_j) = 0, \gamma_i(g) = \chi_i(g)$$

for all $j \in I^+$ and $g \in \Gamma$, and a unique (ε, γ_i) -derivation

$$(3.12) \quad \delta_i : A \rightarrow k \text{ with } \delta_i(a_j) = \lambda_{ji}, \delta_i(g) = 0$$

for all $j \in I^+$ and $g \in \Gamma$. Moreover there is a Hopf algebra map $\Phi : U \rightarrow A^{\text{cop}}$ determined by

$$(3.13) \quad \Phi(z_i) = \gamma_i, \quad \Phi(u_i) = \delta_i$$

for all $i \in I^-$.

Let σ be the 2-cocycle corresponding to Φ (see Section 1). The group-like elements $z_i \otimes g_i^{-1}, i \in I^-$, are central in $(U \otimes A)^\sigma$, and there is an isomorphism of Hopf algebras

$$(3.14) \quad \Psi : U(\mathcal{D}, \lambda) \xrightarrow{\cong} (U \otimes A)^\sigma / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I^-)$$

mapping x_i with $i \in I^-$, x_j with $j \in I^+$, respectively $g \in \Gamma$ onto the residue classes of $u_i \otimes 1, 1 \otimes a_j$, respectively $1 \otimes g$.

PROOF: It can be checked directly as in [2, Lemma 5.19] that the maps γ_i, δ_i and Φ are well-defined by working with the defining relations.

To see that Φ is well-defined without checking the relations we alternatively can apply Theorem 1.2. We define $\varphi : \Lambda \rightarrow \widehat{\Gamma}, (l_i)_{i \in I^-}$, and $s : I^- \rightarrow I^+$ for all $i \in I^-$ by

$$\begin{aligned} \varphi(z_i) &= \chi_i, \\ l_i &= \begin{cases} \lambda_{t(n+k), t(k)} & \text{if } i = t(k) \text{ for some } 1 \leq k \leq n, \\ 0 & \text{otherwise,} \end{cases} \\ s(i) &= \begin{cases} t(n+k) & \text{if } i = t(k) \text{ for some } 1 \leq k \leq n, \\ j_0 & \text{otherwise,} \end{cases} \end{aligned}$$

where j_0 is any element in I^+ .

Hence $l_i \delta_{s(i), j} = \lambda_{ji}$ for all $i \in I^-, j \in I^+$. We have to check the conditions in (1.9). Let $i \in I^-$ with $l_i \neq 0$, that is $i = t(k)$ for some $1 \leq k \leq n$. Then $\varphi(z_{t(k)}) = \chi_{t(k)} = \chi_{t(n+k)}^{-1}$ since $t(k)$ and $t(n+k)$ are linked. This proves the first part of (1.9) since $i = t(k)$ and $s(i) = t(n+k)$. The second part of (1.9) says that $\eta_i(z_j) = \varphi(z_j)(g_{s(i)}) = \chi_j(g_{s(i)})$ for all $j \in I^-$. Since j and $s(i)$ are in different connected components, it follows from the Cartan condition that $\chi_j(g_{s(i)}) = \chi_{s(i)}(g_j)^{-1}$. Since i and $s(i)$ are linked, $\chi_i = \chi_{s(i)}^{-1}$. Hence $\chi_j(g_{s(i)}) = \chi_i(g_j) = \eta_i(z_j)$ by definition of η_i .

The remaining claims of the theorem follow by direct calculations as in [2, Theorem 5.17, end of the proof]. \square

We continue with the situation of Theorem 3.6. Let $\rho \in \widehat{\Lambda}, \chi \in \widehat{\Gamma}$. By [14, Lemma 4.3] the left $(U \otimes A)^\sigma$ -module $L(\rho, \chi)$ is annihilated by all $z_i \otimes g_i^{-1} - 1 \otimes 1, i \in I^-$, if and only if

$$(3.15) \quad \rho(z_i) = \chi(g_i) \text{ for all } i \in I^-.$$

Thus for any $\chi \in \widehat{\Gamma}$ define $\rho \in \widehat{\Lambda}$ by (3.15), and define

$$L(\chi) = L(\rho, \chi)$$

as a left module over $U(\mathcal{D}, \lambda)$ using the isomorphism

$$\Psi : U(\mathcal{D}, \lambda) \xrightarrow{\cong} (U \otimes A)^\sigma / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I^-).$$

We call a character $\chi \in \widehat{\Gamma}$ *dominant* (for \mathcal{D} , λ and I^+) if for all pairs (i, j) of linked elements $1 \leq i, j \leq \theta$, where $i \in I^-, j \in I^+$, there are integers $m_i \geq 0$ such that

$$(3.16) \quad q_{ii}^{m_i} \chi(g_i g_j) = 1.$$

Note that this definition depends on the choice of the sets I^- and I^+ , since $q_{ii} = q_{jj}^{-1}$ for linked pairs (i, j) .

Theorem 3.7. *Let \mathcal{D} be a generic datum of finite Cartan type, and λ a family of linking parameters for \mathcal{D} .*

(1) *The map*

$$\{\chi \in \widehat{\Gamma} \mid \chi \text{ dominant}\} \rightarrow \text{Irr}(U(\mathcal{D}, \lambda)),$$

given by $\chi \mapsto L(\chi)$, is bijective.

(2) *Let L be a finite-dimensional simple left $U(\mathcal{D}, \lambda)$ -module, and $\chi \in \widehat{\Gamma}$. Then $L \cong L(\chi)$ if and only if there is a nonzero element $m \in L$ such that $x_j \cdot m = 0$, for all $j \in I^+$, and $g \cdot m = \chi(g)m$ for all $g \in \Gamma$.*

PROOF: For all $1 \leq i \leq n$ we define

$$z'_i = z_{t(i)}, \eta'_i = \eta_{t(i)}, u'_i = u_{t(i)}, g'_i = g_{t(n+i)}, \chi'_i = \chi_{t(n+i)}, a'_i = a_{t(n+i)}.$$

Let $W' = \bigoplus_{1 \leq i \leq n} k u'_i \in {}^\Lambda \mathcal{YD}$, $V' = \bigoplus_{1 \leq i \leq n} k a'_i \in {}^\Gamma \mathcal{YD}$, and

$$U' = \mathfrak{B}(W') \# k[\Lambda], A' = \mathfrak{B}(V') \# k[\Gamma].$$

We have checked in the proof of Theorem 3.6 that

$$\varphi(z'_i) = (\chi'_i)^{-1} \text{ and } \eta'_i(z) = \varphi(z)(g'_i) \text{ for all } 1 \leq i \leq n, z \in \Lambda.$$

Thus we are in the non-degenerate situation of Section 2 described in (2.1). Let σ' be the corresponding 2-cocycle. Then by Theorem 2.8 the map

$$\{(\rho, \chi) \in \widehat{\Lambda} \times \widehat{\Gamma} \mid (\rho, \chi) \text{ dominant}\} \rightarrow \text{Irr}((U' \otimes A')^{\sigma'}),$$

given by $(\rho, \chi) \mapsto [L_{H'}(\rho, \chi)]$, is bijective, where $H' = (U' \# A')^{\sigma'}$.

Note that $\eta'_i(z'_i) = \eta_{t(i)}(z_{t(i)}) = \chi_{t(i)}(g_{t(i)}) = q_{t(i), t(i)}$. Hence a pair (ρ, χ) is dominant if

$$(3.17) \quad (\eta'_i(z'_i))^{m_i} \rho(z'_i) \chi(g'_i) = q_{t(i), t(i)}^{m_i} \rho(z_{t(i)}) \chi(g_{t(n+i)}) = 1$$

for all $1 \leq i \leq n$, where the m_i are integers ≥ 0 .

Then it follows from Theorem 1.4 that also

$$\{(\rho, \chi) \in \widehat{\Lambda} \times \widehat{\Gamma} \mid (\rho, \chi) \text{ dominant}\} \rightarrow \text{Irr}((U \otimes A)^\sigma),$$

given by $(\rho, \chi) \mapsto [L(\rho, \chi)]$, is bijective.

From [14, Lemma 4.3] and the discussion above we finally obtain that the map

$$\{\chi \in \widehat{\Gamma} \mid \chi \text{ dominant}\} \rightarrow \text{Irr}(U(D, \lambda)),$$

given by $\lambda \mapsto L(\lambda)$, is bijective, where $\chi \in \widehat{\Gamma}$ dominant means that the pair (ρ, χ) with $\rho(z_i) = \chi(g_i)$, $i \in I^-$, is dominant. By (3.17) this latter condition says that $q_{t(i), t(i)}^{m_i} \chi(g_{t(i)} g_{t(n+i)}) = 1$, that is, χ is dominant in the sense of our definition. This proves (1), and (2) follows from Theorem 1.3 (2). \square

For all $1 \leq i \leq 2n$ let $g'_i = g_{t(i)}$, $\chi'_i = \chi_{t(i)}$, $a'_{ij} = a_{t(i), t(j)}$. Then

$$\mathcal{D}' = \mathcal{D}((g'_i)_{1 \leq i \leq 2n}, (\chi'_i)_{1 \leq i \leq 2n}, (a'_{ij})_{1 \leq i, j \leq 2n})$$

is a Cartan datum of finite type. Note that for all $1 \leq i, j \leq n$

$$(3.18) \quad a'_{ij} = a'_{n+i, n+j}, \text{ and } a'_{i, n+j} = 0 = a'_{n+i, j} \text{ for all } 1 \leq i, j \leq n,$$

by Lemma 3.1 (1), and since $t(i) \approx t(n+j)$, $t(j) \approx t(n+i)$. Let $I' = \{1, 2, \dots, 2n\}$. If $i, j \in I'$, then $t(i) \approx t(j)$ in I implies that $i \approx j$ in I' , but in general the converse is not true. We define a family of linking parameters λ' for \mathcal{D}' by

$$\lambda'_{ij} = \begin{cases} \lambda_{t(i), t(j)} & \text{if } t(i) \approx t(j) \text{ in } I, \\ 0 & \text{otherwise} \end{cases},$$

for all $1 \leq i, j \leq 2n$, $i \approx j$ in I' .

There is a surjective Hopf algebra map

$$\pi : U(\mathcal{D}, \lambda) \rightarrow U(\mathcal{D}', \lambda'),$$

given by

$$\pi(x_k) = \begin{cases} x_i, & \text{if } k = t(i), 1 \leq i \leq 2n, \\ 0, & \text{otherwise} \end{cases}$$

and $\pi(g) = g$ for all $g \in \Gamma$. Note that π preserves the linking relations of $U(\mathcal{D}, \lambda)$, since $\lambda_{kl} = 0$ for all k, l with $k \notin t(I')$ or $l \notin t(I')$; and π preserves the Serre relations by (3.18), and since for all $i, j \in I'$ with $t(i) \sim t(j)$ in I and $i \approx j$ in I' the relation $\text{ad}_c(x_i)(x_j) = 0$ follows from the linking relations in $U(\mathcal{D}', \lambda')$.

Corollary 3.8. *The Hopf algebra map $\pi : U(\mathcal{D}, \lambda) \rightarrow U(\mathcal{D}', \lambda')$ induces a bijection*

$$\pi^* : \text{Irr}(U(\mathcal{D}', \lambda')) \rightarrow \text{Irr}(U(\mathcal{D}, \lambda)).$$

PROOF: The isomorphism of Theorem 3.6 defines an isomorphism

$$\Psi' : U(\mathcal{D}', \lambda') \xrightarrow{\cong} (U' \otimes A')^\sigma / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I^-),$$

and the claim follows from the proof of Theorem 3.7. \square

Example 3.9. Suppose that \mathcal{D} is a datum of finite Cartan type such that the Dynkin diagram of the Cartan matrix of \mathcal{D} is the disjoint union of an even number of components of type A_{n_l} , $n_l \geq 2$. Suppose that λ is a family of linking parameters such that the connected components are linked in a circle, the end of one A_{n_l} being linked to the beginning of the next $A_{n_{l+1}}$. Then the Dynkin diagram of \mathcal{D}' in Corollary 3.8 is a union of components of type A_1 , and by Theorem 3.7 and Corollary 2.6 we have an explicit description of the finite-dimensional simple $U(\mathcal{D}, \lambda)$ -modules. \square

3.3. Reduced data of Cartan type. By Corollary 3.8 the computation of the finite-dimensional simple $U(\mathcal{D}, \lambda)$ -modules can be reduced to the non-degenerate case of the $U(\mathcal{D}', \lambda')$ -modules. To describe this case we introduce reduced data.

Definition 3.10. Let Γ be an abelian group, $n \geq 1$, $L_i, K_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ for all $1 \leq i \leq n$, and $(a_{ij})_{1 \leq i, j \leq n}$ a Cartan matrix of finite type. We say that $\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (L_i)_{1 \leq i \leq n}, (K_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n})$ is a *reduced datum of finite Cartan type* if for all $1 \leq i, j \leq n$

$$(3.19) \quad \chi_i(K_j)\chi_j(K_i) = \chi_i(K_i)^{a_{ij}},$$

$$(3.20) \quad \chi_i(L_j) = \chi_j(K_i),$$

$$(3.21) \quad K_i L_i \neq 1, \chi_i(K_i) \neq 1.$$

A reduced datum \mathcal{D}_{red} is called *generic* if $\chi_i(K_i)$ is not a root of unity for all $1 \leq i \leq n$.

Definition 3.11. Let \mathcal{D}_{red} be a reduced datum of finite Cartan type, and $X \in {}_\Gamma \mathcal{YD}$ with basis $x_1, \dots, x_n, y_1, \dots, y_n$, where $x_i \in X_{L_i}^{\chi_i^{-1}}$, $y_i \in X_{K_i}^{\chi_i}$ for all $1 \leq i \leq n$. Let $l = (l_i)_{1 \leq i \leq n}$ be a family of nonzero elements in k . Then we define $U(\mathcal{D}_{red}, l)$ as the quotient Hopf algebra of the smash product $k\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \# k[\Gamma]$ modulo the ideal generated by

$$(3.22) \quad \text{ad}_c(x_i)^{1-a_{ij}}(x_j) \text{ for all } 1 \leq i, j \leq n, i \neq j,$$

$$(3.23) \quad \text{ad}_c(y_i)^{1-a_{ij}}(y_j) \text{ for all } 1 \leq i, j \leq n, i \neq j,$$

$$(3.24) \quad x_i y_j - \chi_j(L_i) y_j x_i - \delta_{ij} l_i (1 - K_i L_i) \text{ for all } 1 \leq i, j \leq n.$$

For all $1 \leq i, j \leq n$ we let $q_{ij} = \chi_j(K_i)$.

The discussion below shows that $U(\mathcal{D}', \lambda') \cong U(\mathcal{D}_{red}, l)$, where \mathcal{D}' and λ' are defined in the end of the last section, and where

$$K_i = g'_i, L_i = g'_{n+i}, \chi_i = \chi'_{n+i}, l_i = \lambda'_{i, n+i}, a_{ij} = a'_{ij}$$

for all $1 \leq i, j \leq n$.

The definition of $U(\mathcal{D}_{red}, l)$ is a special case of Definition 3.4. Indeed define

$$g_i = L_i, g_{n+i} = K_i, \bar{\chi}_i = \chi_i^{-1}, \bar{\chi}_{n+i} = \chi_i \text{ for all } 1 \leq i \leq n,$$

and let $(a_{ij})_{1 \leq i, j \leq 2n}$ be the diagonal block matrix consisting of two identical blocks $(a_{ij})_{1 \leq i, j \leq n}$ on the diagonal. Then

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq 2n}, (\bar{\chi}_i)_{1 \leq i \leq 2n}, (a_{ij})_{1 \leq i, j \leq 2n})$$

is a datum of finite Cartan type. Note that (3.19) and (3.20) together are the Cartan condition (1.1) for \mathcal{D} . We define a family $\lambda = (\lambda_{ij})_{1 \leq i, j \leq 2n, i \not\sim j}$ of linking parameters for \mathcal{D} for all $1 \leq i < j \leq 2n, i \not\sim j$, by

$$\lambda_{ij} = \begin{cases} l_i & \text{if } 1 \leq i \leq n, j = n+i, \\ 0 & \text{otherwise} \end{cases}.$$

The remaining values of λ are determined by (3.5). Thus the Dynkin diagram of \mathcal{D} consists of two copies of the Dynkin diagram of $(a_{ij})_{1 \leq i, j \leq n}$, and each vertex is linked with its copy. Then

$$U(\mathcal{D}_{red}, l) = U(\mathcal{D}, \lambda), \text{ where } y_i = x_{n+i} \text{ for all } 1 \leq i \leq n,$$

since the set of relations (3.22), (3.23), (3.24) coincides with the set (3.7), (3.8).

The linked pairs of vertices of \mathcal{D} are $(i, n+i)$ and $(n+i, i)$, $1 \leq i \leq n$. Hence we can apply Theorem 3.6 to \mathcal{D} with

$$I^- = \{1, 2, \dots, n\}, I^+ = \{n+1, n+2, \dots, 2n\}$$

and $t = \text{id}$. We call a character $\chi \in \hat{\Gamma}$ *dominant* for \mathcal{D}_{red} if there are natural numbers $m_i \geq 0$, $1 \leq i \leq n$, such that

$$(3.25) \quad \chi(K_i L_i) = q_{ii}^{m_i} \text{ for all } 1 \leq i \leq n.$$

Then a character $\chi \in \hat{\Gamma}$ is dominant for \mathcal{D}_{red} if and only χ is dominant for \mathcal{D}, λ and I^+ since for all $1 \leq i \leq n$

$$\bar{\chi}_i(g_i) = \chi_i^{-1}(L_i) = \chi_i^{-1}(K_i) = q_{ii}^{-1}.$$

For $\chi \in \hat{\Gamma}$ we let $L(\chi)$ be the left module over $U(\mathcal{D}_{red}, l) = U(\mathcal{D}, \lambda)$ of Theorem 3.7.

Corollary 3.12. *Let*

$$\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (L_i)_{1 \leq i \leq n}, (K_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n})$$

be a reduced, generic datum of finite Cartan type, and $l = (l_i)_{1 \leq i \leq n}$ be a family of nonzero elements in k .

(1) *The map*

$$\{\chi \in \widehat{\Gamma} \mid \chi \text{ dominant for } U(\mathcal{D}_{red}, l)\} \rightarrow \text{Irr}(U(\mathcal{D}_{red}, l)),$$

given by $\chi \mapsto L(\chi)$, is bijective.

(2) *Let L be a finite-dimensional simple left $U(\mathcal{D}_{red}, l)$ -module, and $\chi \in \widehat{\Gamma}$ a dominant character. Then $L \cong L(\chi)$ if and only if there is a nonzero element $m \in L$ such that $E_i \cdot m = 0$, for all $1 \leq i \leq n$, and $g \cdot m = \chi(g)m$ for all $g \in \Gamma$.*

(3) *Let $\chi \in \widehat{\Gamma}$ a dominant character. Then*

$$L(\chi) = \bigoplus_{\substack{1 \leq i_1, \dots, i_t \leq n, \\ t \geq 0}} L(\chi)^{\chi_{i_1}^{-1} \cdots \chi_{i_t}^{-1} \chi}.$$

PROOF: (1) and (2) follow from Theorem 3.7, and (3) from Proposition 2.2 b). \square

To see that $U_q(\mathfrak{g}), \mathfrak{g}$ a semisimple Lie algebra, and other quantum groups in the literature are special cases of $U(\mathcal{D}_{red}, l)$, we formulate the relations of $U(\mathcal{D}_{red}, l)$ in terms of the usual generators E_i and F_i .

Lemma 3.13. *Let*

$$E_i = y_i, F_i = x_i L_i^{-1}, 1 \leq i, j \leq n.$$

For each connected component $J \in \mathcal{X}$ we choose $q_J \in k$ such that (1.4) holds. For all $1 \leq i, j \leq n$ let

$$p_{ij} = \begin{cases} q_{ij} q_J^{-d_i a_{ij}}, & \text{if } i, j \in J, J \in \mathcal{X}, \\ q_{ij}, & \text{if } i \not\sim j. \end{cases}$$

Then the relations (3.23), (3.22) and (3.24) of $U(\mathcal{D}_{red}, l)$ can be reformulated in $U(\mathcal{D}_{red}, l)$ for all $1 \leq i, j \leq n$ as

$$(3.26) \quad \sum_{s=0}^{1-a_{ij}} (-p_{ij})^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_J^{d_i}} E_i^{1-a_{ij}-s} E_j E_i^s = 0, i \neq j, i \in J, J \in \mathcal{X},$$

$$(3.27) \quad \sum_{s=0}^{1-a_{ij}} (-p_{ij}^{-1})^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_J^{d_i}} F_i^{1-a_{ij}-s} F_j F_i^s = 0, i \neq j, i \in J, J \in \mathcal{X},$$

$$(3.28) \quad E_i F_j - F_j E_i = \delta_{ij} q_{ii}^{-1} l_i (K_i - L_i^{-1}).$$

The action of Γ is given for all $g \in \Gamma$ and all $1 \leq i \leq n$ by

$$(3.29) \quad gE_i g^{-1} = \chi_i(g)E_i, gF_i g^{-1} = \chi_i^{-1}(g)F_i,$$

and the comultiplication by

$$(3.30) \quad \Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i^{-1}.$$

PROOF: This follows from the defining relations of $U(\mathcal{D}_{red}, l)$ and Lemma 3.5. \square

Remark 3.14. In the situation of Corollary 3.12 let \mathcal{M} be the set of all n -tuples $m = (m_i)_{1 \leq i \leq n}$ of integers ≥ 0 for which there exists a dominant character χ for \mathcal{D}_{red} such that $\chi(K_i L_i) = q_{ii}^{m_i}$, $1 \leq i \leq n$. For each $m = (m_i) \in \mathcal{M}$ we choose a dominant character χ_m for \mathcal{D}_{red} satisfying

$$\chi_m(K_i L_i) = q_{ii}^{m_i} \text{ for all } 1 \leq i \leq n.$$

Then any dominant character χ for \mathcal{D}_{red} has the form

$$(3.31) \quad \chi = \psi \chi_m, m \in \mathcal{M}, \psi \in \widehat{\Gamma} \text{ with } \psi(K_i L_i) = 1 \text{ for all } 1 \leq i \leq n,$$

where m and ψ are uniquely determined. Note that the algebra maps $U(\mathcal{D}_{red}, l) \rightarrow k$ are all of the form $\tilde{\psi}$ where $\psi \in \widehat{\Gamma}$ is a dominant character satisfying

$$(3.32) \quad \psi(K_i L_i) = 1 \text{ for all } 1 \leq i \leq n,$$

and where $\tilde{\psi}(E_i) = 0 = \tilde{\psi}(F_i)$ for all $1 \leq i \leq n$, and $\tilde{\psi}(g) = \psi(g)$ for all $g \in \Gamma$. Thus the one-dimensional $U(\mathcal{D}_{red}, l)$ -modules are the modules $L(\psi) = k_{\tilde{\psi}}$ where ψ satisfies (3.32), and it follows from Corollary 3.12 that the finite-dimensional simple $U(\mathcal{D}_{red}, l)$ -modules are of the form

$$(3.33) \quad L(\chi) \cong k_{\tilde{\psi}} \otimes L(\chi_m),$$

where $m \in \mathcal{M}$, and $\psi \in \widehat{\Gamma}$ satisfies (3.32). \square

In the following remarks we discuss several special cases of Corollary 3.12, namely the usual one-parameter deformation $U_q(\mathfrak{g})$ of $U(\mathfrak{g})$, where \mathfrak{g} is a semisimple Lie algebra, Lusztig's version of the one-parameter deformation with more general group-like elements, and two-parameter deformations of $U(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_n)$. In each case the natural choice for the characters χ_m is to define them by dominant weights of the weight lattice of \mathfrak{g} .

Remark 3.15. The quantum group $U_q(\mathfrak{g})$ [11, 4.3], \mathfrak{g} a semisimple Lie algebra, is a special case of $U(\mathcal{D}_{red}, l)$. Let $(a_{ij})_{1 \leq i, j \leq n}$ be the Cartan matrix of \mathfrak{g} with respect to a basis $\alpha_1, \dots, \alpha_n$ of the root system of \mathfrak{g} , $d_i \in \{1, 2, 3\}$ with $d_i a_{ij} = d_j a_{ji}$ for all $1 \leq i, j \leq n$, and $0 \neq q \in k$ such

that $q^{2d_i} \neq 1$ for all $1 \leq i \leq n$. Let Γ be a free abelian group with basis $K_i, 1 \leq i \leq n$, and $L_i = K_i$ or all $1 \leq i \leq n$. Define the characters $\chi_i \in \widehat{\Gamma}$ by

$$\chi_j(K_i) = q^{d_i a_{ij}} \text{ for all } 1 \leq i, j \leq n.$$

Then $\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (L_i), (K_i), (\chi_i), (a_{ij}))$ is a reduced datum, and

$$U(\mathcal{D}_{red}, l) \cong U_q(\mathfrak{g}),$$

where $l_i = q^{2d_i}(q^{d_i} - q^{-d_i})^{-1}$ for all $1 \leq i \leq n$. Note that in Lemma 3.13, $p_{ij} = 1$ for all $1 \leq i, j \leq n$.

Assume that q is not a root of unity, and let $\chi \in \widehat{\Gamma}$. Let χ a dominant character for \mathcal{D}_{red} . By definition there are integers $r_1, \dots, r_n \geq 0$ with $\chi(K_i^2) = q^{2d_i m_i}$, or equivalently

$$\chi(K_i) = \pm q^{d_i m_i} \text{ for all } 1 \leq i \leq n.$$

We say that χ is a character of type 1 if $\chi(K_i) = q^{d_i m_i}$ for all $1 \leq i \leq n$. Thus we see that characters of type 1 correspond uniquely to dominant weights $\lambda = \sum_{i=1}^n m_i \varpi_i$, where $\varpi_1, \dots, \varpi_n$ are the fundamental weights [11, 4.1]. If the dominant weight λ corresponds to the character χ , then

$$L(\lambda) \cong L(\chi),$$

where $L(\lambda)$ is the simple $U_q(\mathfrak{g})$ -module defined in [11, 5.5]. This follows from Corollary 3.12 (2), since $L(\lambda)$ is a finite-dimensional simple $U_q(\mathfrak{g})$ -module and contains a non-zero element v_λ with

$$E_i \cdot v_\lambda = 0, K_i \cdot v_\lambda = q^{(\lambda, \alpha_i)} v_\lambda, 1 \leq i \leq n,$$

where $(\lambda, \alpha_i) = d_i m_i$ (see [11, 4.1]) for all $1 \leq i \leq n$. \square

Remark 3.16. Similarly the quantum group \mathbf{U} defined in [9, 3.1.1], where the Cartan datum (I, \cdot) is of finite type, is a special case of $U(\mathcal{D}_{red}, l)$ (This also holds in general, if we assume that (a_{ij}) is a symmetrizable generalized Cartan matrix). In [9, 3.1.1] a root datum $(Y, X, <, >, \dots)$ of type (I, \cdot) is given. Let Γ be the group with generators $K_\mu, \mu \in Y$, and relations $K_0 = 1, K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in Y$. Thus $\Gamma \cong Y$ is a free abelian group of finite rank. To avoid confusion we denote the elements of Γ by K'_μ and not by K_μ . Define characters $\chi_i \in \widehat{\Gamma}$ by

$$\chi_j(K'_\mu) = v^{<\mu, j'>} \text{ for all } \mu \in Y, j \in I.$$

For all $i, j \in I$ we define $d_i = \frac{i \cdot i}{2}, a_{ij} = <i, j'>$, and $K_i = L_i = K'_{d_i i}$. Then (a_{ij}) is a Cartan matrix of finite type, and

$$\chi_j(K_i) = v^{d_i a_{ij}} \text{ for all } i, j \in I.$$

Again $\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (L_i), (K_i), (\chi_i), (a_{ij}))$ is a reduced datum (where we identify I with $\{1, 2, \dots, n\}$), and

$$U(\mathcal{D}_{red}, l) \cong U_q(\mathfrak{g}),$$

where $l_i = v^{2d_i}(v^{d_i} - v^{-d_i})^{-1}$ for all $1 \leq i \leq n$.

Note that the matrix $(\chi_j(K_i))$ is symmetric as in the previous remark, but it is not assumed that the elements K_i generate the group Γ .

Let M be a left \mathbf{U} -module. For any $\lambda \in X$ let

$$M^\lambda = \{m \in M \mid K'_\mu m = v^{<\mu, \lambda>} m\}$$

be the weight space of M of weight λ . The category \mathcal{C} consists of all left \mathbf{U} -modules M such that

$$M = \bigoplus_{\lambda \in X} M^\lambda.$$

By [9, 3.5.5, 6.3.4] the finite-dimensional simple \mathbf{U} -modules in \mathcal{C} are isomorphic to Λ_λ , where $\lambda \in X$ is dominant, that is $<i, \lambda> \in \mathbb{N}$ for all $i \in I$. Let $\lambda \in X$ be dominant. The \mathbf{U} -module Λ_λ contains a nonzero element η_λ of weight λ such that $E_i \cdot \eta_\lambda = 0$ for all $i \in I$. By Corollary 3.12, $\Lambda_\lambda \cong L(\chi)$, where $\chi \in \widehat{\Gamma}$ is defined by $\chi(K'_\mu) = v^{<\mu, \lambda>}$ for all $\mu \in Y$. Note that $\chi(K_i) = v^{d_i m_i}$ for all $i \in I$, where $m_i = <i, \lambda> \in \mathbb{N}$.

Thus we recover the classification of finite-dimensional simple \mathbf{U} -modules in the category \mathcal{C} . In Corollary 3.12 and (3.33) this classification is extended to *all* finite-dimensional simple \mathbf{U} -modules. \square

Remark 3.17. In [6] Benkart and Witherspoon give a classification of the finite-dimensional simple modules over a two-parameter deformation of the general linear and special linear Lie algebras. We will see that their classification in [6, Theorem 2.19] is a special case of Corollary 3.12. Fix a natural number $n \geq 1$, nonzero elements $r, s \in k$, and assume that rs^{-1} is not a root of unity. Let E be a euclidean vector space with inner product $(,)$ and orthonormal basis $\epsilon_1, \dots, \epsilon_{n+1}$. Define $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for all $1 \leq i \leq n$. Thus $\{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n+1, i \neq j\}$ is a root system of type A_n with basis $\alpha_1, \dots, \alpha_n$.

Let $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ be the basis of a free abelian group of rank $2(n+1)$. Define characters $\chi_j \in \widehat{\Gamma}$ for all $1 \leq j \leq n$ by

$$\chi_j(a_i) = r^{(\epsilon_i, \alpha_j)}, \chi_j(b_i) = s^{(\epsilon_i, \alpha_j)} \text{ for all } 1 \leq i \leq n+1.$$

Let $K_i = a_i b_{i+1}$, $L_i = (a_{i+1} b_i)^{-1}$ and $q_{ij} = \chi_j(K_i)$ for all $1 \leq i, j \leq n$. Then for all $1 \leq i, j \leq n$,

$$\begin{aligned}\chi_j(K_i) &= r^{(\epsilon_i, \alpha_j)} s^{(\epsilon_{i+1}, \alpha_j)} = \chi_i(L_j), \\ q_{ii} &= r s^{-1}, \\ q_{ij} q_{ji} &= \begin{cases} q_{ii}^{-1}, & \text{if } |i - j| = 1, \\ 1, & \text{if } |i - j| > 1. \end{cases}\end{aligned}$$

Thus the Cartan condition (1.1) is satisfied with the Cartan matrix $(a_{ij})_{1 \leq i, j \leq n}$ of type A_n defined by

$$a_{ij} = \begin{cases} -1, & \text{if } |i - j| = 1, \\ 0, & \text{if } |i - j| > 1, \end{cases}$$

and $\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (L_i), (K_i), (\chi_i), (a_{ij}))$ is a reduced datum. Note that the matrix (q_{ij}) is not symmetric. Choose $q \in k$ with $q^2 = r s^{-1}$. Then the Serre relations (3.26) for $|i - j| = 1$ are here

$$E_i^2 E_j - p_{ij}(q + q^{-1}) E_i E_j E_i + p_{ij}^2 E_j E_i^2 = 0.$$

Since $q_{i,i+1} = s$, $q_{i+1,i} = r^{-1}$ we find

$$\begin{aligned}p_{i,i+1}(q + q^{-1}) &= r + s, & p_{i,i+1}^2 &= r s, \\ p_{i+1,i}(q + q^{-1}) &= r^{-1} + s^{-1}, & p_{i+1,i}^2 &= r^{-1} s^{-1},\end{aligned}$$

hence

$$\begin{aligned}E_i^2 E_{i+1} - (r + s) E_i E_{i+1} E_i + r s E_{i+1} E_i^2 &= 0, \\ E_i E_{i+1}^2 - (r + s) E_{i+1} E_i E_{i+1} + r s E_{i+1}^2 E_i &= 0.\end{aligned}$$

The Serre relations (3.27) for $|i - j| = 1$ are

$$\begin{aligned}F_i^2 F_{i+1} - (r^{-1} + s^{-1}) F_i F_{i+1} F_i + r^{-1} s^{-1} F_{i+1} F_i^2 &= 0, \\ F_i F_{i+1}^2 - (r^{-1} + s^{-1}) F_{i+1} F_i F_{i+1} + r^{-1} s^{-1} F_{i+1}^2 F_i &= 0.\end{aligned}$$

Thus we have established the relations (R6) and (R7) in [6], and

$$U(\mathcal{D}_{red}, l) = U_{r,s}(\mathfrak{gl}_{n+1}),$$

defined in [6], with $E_i = e_i$, $F_i = f_i$ for all $1 \leq i \leq n$, where we take $l_i = r s^{-1}(r - s)^{-1}$ for all $1 \leq i \leq n$, and $l = (l_i)$.

A dominant weight λ is an element λ in $\mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_{n+1}$ such that $(\alpha_i, \lambda) \in \mathbb{N}$ for all $1 \leq i \leq n$. If $\lambda = \sum_{i=1}^{n+1} \lambda_i \epsilon_i$, where $\lambda_i \in \mathbb{Z}$ for all $1 \leq i \leq n + 1$, then λ is dominant if and only if

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}.$$

Following [6] we define for any dominant weight λ a character $\widehat{\lambda}$ by

$$\widehat{\lambda}(a_i) = r^{(\epsilon_i, \lambda)}, \widehat{\lambda}(b_i) = s^{(\epsilon_i, \lambda)} \text{ for all } 1 \leq i \leq n+1.$$

Note that

$$\widehat{\lambda}(K_i L_i) = (rs^{-1})^{(\alpha_i, \lambda)} = q_{ii}^{(\alpha_i, \lambda)} \text{ for all } 1 \leq i \leq n+1.$$

Thus $\widehat{\lambda}$ is a dominant character in our sense. If $m = (m_i)_{1 \leq i \leq n}$ is any family of natural numbers ≥ 0 , we define $\lambda_i = \sum_{k=i}^n m_k$, $1 \leq i \leq n$, and $\lambda_{n+1} = 0$. Then $m_i = (\alpha_i, \lambda)$ for all $1 \leq i \leq n$, where $\lambda = \sum_{i=1}^{n+1} \lambda_i \epsilon_i$.

This shows that the classification of the finite-dimensional simple $U_{r,s}(\mathfrak{gl}_{n+1})$ -modules (and similarly $U_{r,s}(\mathfrak{sl}_{n+1})$ -modules) in [6, Theorem 2.19] is a special case of our Corollary 3.12. \square

3.4. The Finite-dimensional Case. In this section we fix a finite abelian group Γ , a datum $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta})$ of finite Cartan type, and a family λ of linking parameters for \mathcal{D} .

We assume that for all $1 \leq i \leq \theta$, $\text{ord}(q_{ii})$ is odd and > 3 , that $\text{ord}(q_{ii})$ is prime to 3 if i is in a component G_2 .

For each connected component $J \in \mathcal{X}$, and positive root α of the root system Φ_J of J let x_α be the root vector in the free algebra $k\langle x_j \mid j \in J \rangle$ defined in [2, Section 4.1] generalizing the root vectors in [9].

Definition 3.18. Let $u(\mathcal{D}, \lambda)$ be the quotient Hopf algebra of the smash product $k\langle x_1, \dots, x_\theta \rangle \# k[\Gamma]$ modulo the ideal generated by (3.7), (3.8) and

$$(3.34) \quad x_\alpha^{N_J} \text{ for all } \alpha \in \Phi_J^+, J \in \mathcal{X}.$$

In addition we assume that *the linking graph of \mathcal{D}, λ is bipartite*. By Lemma 3.2 this holds in particular if (a_{ij}) is simply laced.

We proceed exactly as in the previous section and use the above notations. The only difference is the definition of Λ as in [2, proof of Theorem 5.17].

Let Λ be the abelian group with generators $z_i, i \in I^-$, and relations $z_i^{n_i} = 1$ for all $i \in I^-$, where n_i is the least common multiple of $\text{ord}(g_i)$ and $\text{ord}(\chi_i)$. For all $j \in I^-$ let $\eta_j \in \widehat{\Lambda}$ be defined by $\eta_j(z_i) = \chi_j(g_i)$ for all $i \in I^-$.

Let $W \in {}^\Lambda \mathcal{YD}$ with basis $u_i \in W_{z_i}^{\eta_i}, i \in I^-$, and $V \in {}^\Gamma \mathcal{YD}$ with basis $a_j \in V_{g_j}^{\chi_j}, j \in I^+$. We define

$$U = \mathfrak{B}(W) \# k[\lambda] \text{ and } A = \mathfrak{B}(V) \# k[\lambda].$$

Then $U = \mathfrak{B}(W)$ and $A = \mathfrak{B}(V)$ by [2, Theorem 4.5].

As in the infinite case using Theorem 1.2 we define for all $i \in I^-$ the algebra map $\gamma_i : A \rightarrow k$, the (ε, γ_i) -derivation $\delta_i : A \rightarrow k$ and the Hopf

algebra map $\Phi : U \rightarrow A^{0\text{cop}}$. Let σ be the 2-cocycle corresponding to Φ . As in the proof of [2, Theorem 5.17] we obtain

Theorem 3.19. *The group-like elements $z_i \otimes g_i^{-1}$, $i \in X_2$, are central in $(U \otimes A)^\sigma$, and there is an isomorphism of Hopf algebras*

$$u(\mathcal{D}, \lambda) \cong (U \otimes A)^\sigma / (z_i \otimes g_i^{-1} - 1 \otimes 1 \mid i \in I^-)$$

mapping x_i with $i \in I^-$, x_j with $j \in I^+$, resp. $g \in \Gamma$ onto the residue classes of $u_i \otimes 1$, $1 \otimes a_j$ resp. $1 \otimes g$.

For any $\chi \in \hat{\Gamma}$ define $\rho \in \hat{\Lambda}$ by $\rho(z_i) = \chi(g_i)$ for all $i \in I^-$. As above we define $L(\chi) = L(\chi, \rho)$ as a left module over $u(\mathcal{D}, \lambda)$. As in the infinite case we define the reduced datum \mathcal{D}' , and $u(\mathcal{D}', \lambda')$ by adding the root vector relations, and the Hopf algebra projection $F : u(\mathcal{D}, \lambda) \rightarrow u(\mathcal{D}', \lambda')$.

Theorem 3.20. *The function*

$$\hat{\Gamma} \rightarrow \text{Irr}(u(\mathcal{D}, \lambda)), \chi \mapsto [L(\chi)],$$

is bijective, and pullback along F defines a bijection

$$F^* : \text{Irr}(u(\mathcal{D}', \lambda')) \rightarrow \text{Irr}(u(\mathcal{D}, \lambda)).$$

PROOF: This is shown as in the proof of Theorem 3.7 and Corollary 3.8 using Theorem 1.1 (2). \square

In particular each $x_k, k \neq t(i), 1 \leq i \leq 2n$, is contained in the kernel of F , hence x_k lies in the Jacobson radical of $u(\mathcal{D}, \lambda)$ by Theorem 3.20. We give another proof of this fact without using the bipartiteness assumption.

Theorem 3.21. *Let \mathcal{D}, λ be as in the beginning of this section but where the linking graph of \mathcal{D}, λ is not necessarily bipartite. Let $1 \leq k \leq \theta$ be a vertex which is not linked to any other vertex. Then x_k is contained in the Jacobson radical of $u(\mathcal{D}, \lambda)$.*

PROOF: Let J be the connected component of the Dynkin diagram containing k . Let u_J be the subalgebra of $u(\mathcal{D}, \lambda)$ generated by all $x_j, j \in J$, and let u' be the subalgebra of $u(\mathcal{D}, \lambda)$ generated by all $g \in \Gamma$ and $x_l, l \notin J$. Then using the PBW-basis in [5, Theorem 3.3] it follows that

$$u(\mathcal{D}, \lambda) = u' u_J.$$

Since i is not linked, x_k skew-commutes with all the generators of u' . Hence $u(\mathcal{D}, \lambda) x_k u(\mathcal{D}, \lambda) \subseteq u(\mathcal{D}, \lambda) u_J^+$. Since the augmentation ideal u_J^+ of u_J is nilpotent we see that x_k generates a nilpotent ideal in $u(\mathcal{D}, \lambda)$.

\square

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